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PARTITIONS IN TERNARS

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A *ternar* is defined as a quintuple $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ where S_1, S_2, S_3, S_0 are nonempty sets and τ is a mapping of $S_1 \times S_2 \times S_3$ into S_0. By $\lambda(\mathbf{T})$ we denote the set of all $(u, v) \in S_3 \times S_0$ for which at least one $(x, y) \in S_1 \times S_2$ exist such that $\tau(x, y, u) = v$.

Let a ternar $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ be given. We shall investigate the following conditions:

(1) There exist an element $o \in S_3$ and an injection $\xi : S_2 \to S_0$ such that $\tau(x, y, o) = \xi(y)$ holds for all $(x, y) \in S_1 \times S_2$.

(2) Every equation $\tau(x, y, u) = v$ has a unique solution $x \in S_1$ for each triple $(y, u, v) \in S_2 \times S_3 \times S_0$.

(3) Every equation $\tau(x_1, y_1, u) = \tau(x_2, y_2, u)$ has a unique solution $u \in S_3$ for any distinct pairs $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$.

(4) Every pair of equations $\tau(x, y, u_i) = v_i$ $(i = 1, 2)$ has a unique solution $(x, y) \in S_1 \times S_2$ for any $(u_1, v_1), (u_2, v_2) \in \lambda(\mathbf{T})$ with distinct u_1, u_2.

The geometric meaning of conditions (1)—(4) can be described as follows.

Proposition 1. *Let $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ be a ternar. We shall call the pairs $(x, y) \in S_1 \times S_2$ the "points" and the sets $L(u, v) = \{(x, y) \mid \tau(x, y, u) = v\}$, $(u, v) \in \lambda(\mathbf{T})$ will be termed the "lines". Then condition (i) is equivalent with condition (i'), $i = 1, 2, 3, 4$, where:*

(1') There exist an element $o \in S_3$ and an injection $\xi : S_2 \to S_0$ such that $L(o, \xi(y)) = \{(x, y) \mid x \in S_1\}$ for all $y \in S_2$.

(2') For all $(c, u, v) \in S_2 \times S_3 \times S_0$, the intersection of $L(u, v)$ and $\{(x, y) \mid y = c\}$ contains exactly one point.

(3') To any two distinct points, there exists exactly one pair $(u, v) \in \lambda(\mathbf{T})$ such that the line $L(u, v)$ contains the given points.

(4') Any two lines $L(u_1, v_1), L(u_2, v_2)$ with $u_1 \neq u_2$ intersect in exactly one point.

The proof is omitted.

By a *partition in* a ternar $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ is meant a quadruple $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ where $\mathscr{P}_i$ is a partition in S_i for $i = 1, 2, 3, 0$; [1], p. 42 and [2], p. 14, respectively. If, in particular, $\mathscr{P}_i$ is a partition on S_i for $i = 1, 2, 3$ then $\mathbf{P}$ is said to be a *partition on* $\mathbf{T}$. If, for each $(P_1, P_2, P_3) \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$, a $P_0 \in \mathscr{P}_0$ exists with $\tau(P_1, P_2, P_3) \subseteq P_0$ then

P is said to be *generating*. If **P** is a generating partition in **T** then a factorternar $\mathbf{T}/\mathbf{P} = (P_1, P_2, P_3, P_0, \tau/\mathbf{P})$ is well-defined with $\tau(P_1, P_2, P_3) \subseteq \tau/\mathbf{P}\,(P_1, P_2, P_3) \in \mathscr{P}_0$ for each $(P_1, P_2, P_3) \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$. A generating partition **P** in **T** is called *(i)-permitting* if **T**/**P** satisfies the condition (i) where $i = 1, 2, 3, 4$.

If $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ and $\mathbf{Q} = (\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_0)$ are partitions in a ternar $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ one defines $\sup(\mathbf{P}, \mathbf{Q}) = (\sup(\mathscr{P}_1, \mathscr{Q}_1), \sup(\mathscr{P}_2, \mathscr{Q}_2), \sup(\mathscr{P}_3, \mathscr{Q}_3), \sup(\mathscr{P}_0, \mathscr{Q}_0)$ and $\inf(\mathbf{P}\ \mathbf{Q}) = (\inf(\mathscr{P}_1, \mathscr{Q}_1), \inf(\mathscr{P}_2, \mathscr{Q}_2), \inf(\mathscr{P}_3, \mathscr{Q}_3), \inf(\mathscr{P}_0, \mathscr{Q}_0))$ where, in the second case, the existence of infima on the right side must be supposed; for the notion of supremum and infimum, cf. [1], pp. 43—45 and [2], pp. 15—18.

In the sequel we shall find some algebraic properties of generating or (i)-permitting partitions in a ternar $(i = 1, 2, 3, 4)$.

Propositon 2. *Let* $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ *and* $\mathbf{Q} = (\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_0)$ *be generating partitions in a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$. a) *If* inf (**P**, **Q**) *exists then it is generating too.* b) *If* **P** *and* **Q** *are generating partitions on* **T** *then* sup (**P**, **Q**) *is generating too.* c) *For generating partitions* **P**, **Q** *in* **T**, sup (**P**, **Q**) *is, in general, not generating.*

Proof. a) Let $P_i \cap Q_i$ be an arbitrary element of $\inf(\mathscr{P}_i, \mathscr{Q}_i)$ where $P_i \in \mathscr{P}_i$ and $Q_i \in \mathscr{Q}_i$; $i = 1, 2, 3$. Then $\tau(P_1, P_2, P_3) \subseteq P_0$ and $\tau(Q_1, Q_2, Q_3) \subseteq Q_0$ for uniquely determined elements $P_0 \in \mathscr{P}_0$, $Q_0 \in \mathscr{Q}_0$ so that $\tau(P_1 \cap Q_1, P_2 \cap Q_2, P_3 \cap Q_3) \subseteq P_0 \cap Q_0$. Thus inf (**P**, **Q**) must be generating.

b) Let **P** and **Q** be generating on **T**. Let A_i, B_i be two elements of $\mathscr{P}_i$ which belong to the same element of $\sup(\mathscr{P}_i, \mathscr{Q}_i)$; $i = 1, 2, 3$. Consequently, for each $i = 1, 2, 3$ there exists a *"chaining sequence"* $A_i = A_0^i, C_0^i, A_1^i, C_1^i, \ldots, A_{r_i}^i = B_i$ with $A_j^i \in \mathscr{P}_i$ and $C_j^i \in \mathscr{Q}_i$, where every two consecutive members have a nonempty intersection. Without loosing generality, we may suppose that $r_1 = r_2 = r_3 = r$. To each triple (A_j^1, A_j^2, A_j^3) and (C_j^1, C_j^2, C_j^3), respectively, there exists an element $A_j^0 \in \mathscr{P}_0$ and $C_j^0 \in \mathscr{Q}_0$, respectively, such that $\tau(A_j^1, A_j^2, A_j^3) \subseteq A_j^0$ and $\tau(C_j^1, C_j^2, C_j^3) \subseteq C_j^0$. From this it follows that A_0^0 and A_r^0 lie in the same element of $\sup(\mathscr{P}_0, \mathscr{Q}_0)$. Consequently, for arbitrarily given elements $D_i \in \sup(\mathscr{P}_i, \mathscr{Q}_i)$, $i = 1, 2, 3$, there exists an element $D_0 \in \sup(\mathscr{P}_0, \mathscr{Q}_0)$ such that $\tau(D_1, D_2, D_3) \subseteq D_0$. Thus sup (**P**, **Q**) must be generating.

c) Choose a ternar $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ such that $S_1 = \{a, b\}$, $S_2 = \{c, d\}$, $S_3 = \{e\}$, $S_0 = \{f, g, h, k\}$ and $\tau(a, c, e) = f$, $\tau(a, d, e) = g$, $\tau(b, c, e) = h$, $\tau(b, d, e) = k$. Further, let $\mathscr{P}_1 = \{\{a\}\}$, $\mathscr{P}_2 = \{\{c\}\}$, $\mathscr{P}_3 = \{\{e\}\}$, $\mathscr{P}_0 = \{\{f\}\}$ and $\mathscr{Q}_1 = \{\{b\}\}$, $\mathscr{Q}_2 = \{\{d\}\}$, $\mathscr{Q}_3 = \{\{e\}\}$, $\mathscr{Q}_0 = \{\{k\}\}$. Then $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ and $\mathbf{Q} = (\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_0)$ are generating but $\sup(\mathbf{P}, \mathbf{Q}) = (\{\{a\}, \{b\}\}, \{\{c\}, \{d\}\}, \{\{e\}\}, \{\{f\}, \{k\}\})$ is not generating because of $\tau(a, d, e) = g$ and $\tau(b, c, e) = h$.

Proposition 3. *Let* $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ *and* $\mathbf{Q} = (\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_0)$ *be generating partitions in a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$. *If*

$$(5) \qquad \bigcup_{P_i \in \mathscr{P}_i} P_i = \bigcup_{Q_i \in \mathscr{Q}_i} Q_i \text{ for } i = 1, 2, 3$$

then inf $(\mathbf{P}, \mathbf{Q})$ *and* sup $(\mathbf{P}, \mathbf{Q})$ *are generating too.*

The proof follows by modifying of the proof of Proposition 2ab.

Note. A *n-nar* can be defined as a sequence $\mathbf{T} = (S_1, \ldots, S_n, S_0, \tau)$ where $S_1, \ldots, S_n, S_0$ are nonempty sets and τ a mapping of $S_1 \times \ldots \times S_n$ into S_0. The notion of a partition in (on) $\mathbf{T}$ or of a generating partition in (on) $\mathbf{T}$ can be introduced analogously as by ternars. Then Propositions 2—3 and the idea of their proofs remain valid also in the case if $\mathbf{T}$ is a n-nar and $\mathbf{P}$, $\mathbf{Q}$ are generating partitions in $\mathbf{T}$.

Proposition 4. *Let a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ *satisfy condition* (1). *Let* $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ *be a generating partition in* $\mathbf{T}$. *Then* $\mathbf{P}$ *is* (1)-*permitting.* Proof. Let $O \in \mathscr{P}_3$ contain the element $o \in S_3$. For each pair $(X, Y) \in \in \mathscr{P}_1 \times \mathscr{P}_2$, one obtains $\{\xi(y) \mid y \in Y\} \subseteq \tau(X, Y, O) \subseteq \tau/\mathbf{P}(X, Y, O)$. Thus we can define an injection $\xi/\mathbf{P} : \mathscr{P}_2 \to \mathscr{P}_0$ in such a way that $\xi/\mathbf{P}(Y) = \xi/\mathbf{P}(X, Y, O)$.

Proposition 5. *Let* $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ *be a* (2)-*permitting partition on a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$. *Let* $\mathbf{P}$ *satisfy the condition* (2). *Then* $\tau(P_1, P_2, P_3) \in \mathscr{P}_0$ *for all triples* $(P_1, P_2, P_3) \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$.

Proof. To each triple $(P_1, P_2, P_3) \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$, there exists exactly one $P_0 \in \mathscr{P}_0$ such that $\tau(P_1, P_2, P_3) \subseteq P_0$. But for each $(p_2, p_3, p_0) \in \in P_2 \times P_3 \times P_0$ there exists exactly one $p_1 \in S_1$ such that $\tau(p_1, p_2, p_3) = = p_0$. If $p_1 \in P_1' \in \mathscr{P}_1$, $P_1' \neq P_1$ then $\tau(P_1', P_2, P_2) \subseteq P_0$ and consequently, $P_1' = P_1$ by the assumption that $\mathbf{P}$ is (2)-permitting. This contradiction finishes the proof.

Proposition 6. *Let* $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ *be a* (2)-*permitting partition in a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$. *Suppose that for each* $(P_1, P_2, P_3, P_0) \in \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3 \times \mathscr{P}_0$ *with* $P_0 \subseteq \tau(P_1, P_2, P_3)$, *the following condition is fulfilled:*

(6) To each $p_0 \in P_0$, there exists a $(p_1, p_2, p_3) \in P_1 \times P_2 \times P_3$ such that $\tau(p_1, p_2, p_3) = p_0$.

Then $\tau(P_1, P_2, P_3) \in \mathscr{P}_0$ *for each triple* $(P_1, P_2, P_3) \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$. The proof follows by a slight modification of the proof of Proposition 5.

Proposition 7. *Let* $\mathbf{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_0)$ *and* $\mathbf{Q} = (\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_0)$ *be* (2)-*permitting partitions in a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ *which satisfies condition* (2). a) *Suppose that inf* $(\mathbf{P}, \mathbf{Q})$ *exists. Then it is* (2)-*permitting too.* b) *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are partitions on* $\mathbf{T}$ *then sup* $(\mathbf{P}, \mathbf{Q})$ *is* (2)-*permitting too.*

Proof. a) Let $P_j \cap Q_j \in \inf(\mathscr{P}_j, Q_j); j = 2, 3, 0$. Then there exist uniquely determined elements $P_1 \in \mathscr{P}_1$ and $Q_1 \in \mathscr{Q}_1$ satisfying $\tau/\mathbf{P}(P_1, P_2, P_3) = P_0$

and $\tau/\mathcal{Q}(Q_1, Q_2, Q_3) = Q_0$, respectively. From this it follows that there is exactly one element $P_1 \cap Q_1 \in \inf(\mathcal{P}_1, \mathcal{Q}_1)$ such that $\tau/\inf(\mathbf{P}, \mathbf{Q})(P_1 \cap Q_1, P_2 \cap Q_2, P_3 \cap Q_3) = P_0 \cap Q_0$ so that $\inf(\mathbf{P}, \mathbf{Q})$ is (2)-permitting.

b) We shall show that, for arbitrarily given $(Y, U, V) \in \sup(\mathcal{P}_2, \mathcal{Q}_2) \times \sup(\mathcal{P}_3 \times \mathcal{Q}_3) \times \sup(\mathcal{P}_0, \mathcal{Q}_0)$, there is a unique $X \in \sup(\mathcal{P}_1, \mathcal{Q}_1)$ such that $\tau/\sup(\mathbf{P}, \mathbf{Q})(X, Y, U) = V$. Choose arbitrary elements $P_i^j \in P_j$; $i = 1, 2$; $j = 2, 3, 0$ and suppose that P_1^j and P_2^j lie simultaneously in the same element of $\sup(\mathcal{P}_j, \mathcal{Q}_j)$; $j = 2, 3, 0$. Thus, for $j = 2, 3, 0$, there exists a chaining sequence $P_1^j = A_0^j, B_0^j, A_1^j, B_1^j, \ldots A_{r_j}^j = P_2^j$ where $A_k^j \in \mathcal{P}_j$, $B_k^j \in \mathcal{L}_j$ and $A_k^j \cap B_k^j \neq 0 \neq A_{k+1}^j \cap B_k^j$ $(k = 0, 1, \ldots, r_{j-1})$. Without loosing generality we may suppose that $r_2 = r_3 = r_0 = r$. Further, we find uniquely determined elements $A_0^1, B_0^1, \ldots, A_r^1$ of $\mathcal{P}_1$ or $\mathcal{Q}_1$, respectively, satisfying $\tau/\mathbf{P}(A_k^1, A_k^2, A_k^3) = A_k^0$ or $\tau/\mathbf{Q}(B_k^1, B_k^2, B_k^3) = B_k^0$, $k = 0, 1, \ldots, r$. From this it follows that $A_0^1, B_0^1, \ldots, A_r^1$ is a chaining sequence between A_0^1 and A_r^1 (any two consecutive membres must intersect) so that A_0^1 and A_r^1 belong to the same element of $\sup(\mathcal{P}_1, \mathcal{Q}_1)$. Consequently, $\sup(\mathbf{P}, \mathbf{Q})$ must be (2)-permitting.

Proposition 8. *Let* $\mathbf{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_0)$ *and* $\mathbf{Q} = (\mathcal{Q}, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_0)$ *be two* (3)*-permitting partitions in a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ *satisfying condition* (3). a) *If there exists* $\inf(\mathbf{P}, \mathbf{Q})$ *then it is* (3)*-permitting too.* b) *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are partitions on* $\mathbf{T}$ *then* $\sup(\mathbf{P}, \mathbf{Q})$ *is* (3)*-permitting too.*

Proof. a) We shall show that the equation $\tau/\inf(\mathbf{P}, \mathbf{Q})(X_1, Y_1, U) = \tau/\inf(\mathbf{P}, \mathbf{Q})(X_2, Y_2, U)$ has a unique solution $U \in \inf(\mathcal{P}_3, \mathcal{Q}_3)$ for arbitrarily given distinct $(X_1, Y_1), (X_2, Y_2) \in \inf(\mathcal{P}_1, \mathcal{Q}_1) \times \inf(\mathcal{P}_2, \mathcal{Q}_2)$. The elements X_1, Y_1, X_2, Y_2 have the forms $P_1^1 \cap Q_1^1$, $P_2^1 \cap Q_2^1$, $P_1^2 \cap Q_1^2$, $P_2^2 \cap Q_2^2$, where $P_i^j \in \mathcal{P}_j$ and $Q_i^j \in \mathcal{Q}_i$ for $i, j = 1, 2$. There exist uniquely determined elements $U_1 \in \mathcal{P}_3$, $U_2 \in \mathcal{Q}_3$ satisfying $\tau/\mathbf{P}(P_1^1, P_2^1, U_1) = \tau/\mathbf{P}(P_1^2, P_2^2, U_1)$ or $\tau/\mathbf{Q}(Q_1^1, Q_2^1, U_2) = \tau/\mathbf{Q}(Q_1^2, Q_2^2, U_2)$, respectively. The starting equation has a unique solution $U_1 \cap U_2 \in \inf(\mathcal{P}_3, \mathcal{Q}_3)$.

b) We have to show that the equation $\tau/\sup(\mathbf{P}, \mathbf{Q})(X_1, Y_1, U) = \tau/\sup(\mathbf{P}, \mathbf{Q})(X_2, Y_2, U)$ has precisely one solution $U \in \sup(\mathcal{P}_3, \mathcal{Q}_3)$ for arbitrarily given distinct $(X_1, Y_1), (X_2, Y_2) \in \sup(\mathcal{P}_1, \mathcal{Q}_1) \times \sup(\mathcal{P}_2, \mathcal{Q}_2)$. It suffices to prove that for any $A_i, B_i \in \mathcal{P}_1$ contained in X_i and for any $C_i, D_i \in \mathcal{P}_2$ contained in Y_i, the elements $E_1, E_2 \in \mathcal{P}_3$ determined uniquely by $\tau/\mathbf{P}(A_i, C_i, E_i) = \tau/\mathbf{P}(B_i, D_i, E_i)$; $i = 1,2$ lie in the same element of $\sup(\mathcal{P}_3, \mathcal{Q}_3)$. But this result follows from the existence of chaining sequences of common length between A_1 and A_2, C_1 and C_2, B_1 and B_2, D_1 and D_2, respectively, analogously to the proof of Proposition 6b.

Proposition 9. *Let* $\mathbf{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_0)$ *be a partition on a ternar* $\mathbf{T} = (S_1, S_2, S_3, S_0, \tau)$ *which satisfies condition* (4). *Let there exist pairs*

$(u_1, v_1), (u_2, v_2) \in \lambda(\mathbf{T})$ such that u_1, u_2 are distinct elements of the same element of $\mathscr{P}_3$ whereas v_1 and v_2 have to belong to distinct elements of $\mathscr{P}_0$. Then $\mathbf{P}$ is not generating.

Proof. The assumptions state that, for $i = 1, 2$, $u_i \in U \in \mathscr{P}_3$ and $v_i \in V_i \in \mathscr{P}_0$ where $u_1 \neq u_2$ and $V_1 \neq V_2$. By (4), there is exactly one $(x, y) \in S_1 \times S_2$ such that $\tau(x, y, u_i) = v_i$ for $i = 1, 2$. Let $X \in \mathscr{P}_1$ contain x and let $Y \in \mathscr{P}_2$ contain y. Thus $\tau(X, Y, U)$ contains an element of V_1 and simultaneously an element of $V_2 \neq V_1$ so that $\mathbf{P}$ does not be generating.

LITERATURE

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