C. Tudor

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A FIXED POINT THEOREM IN LOCALLY CONVEX SPACES

By G. Tudor, Bucharest

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In this paper there is extended a fixed point theorem obtained in [1] to locally convex spaces. Therefore, let $E$ be a Hausdorff locally convex space and $\mathcal{A}$ a sufficiently and directed family of seminorms that gives the topology of $E$.

Let $\varphi$ be a mapping of the family $\mathcal{A}$ satisfying the condition

\[(1) \quad \varphi[\varphi(x)] = \varphi(x) \quad (\alpha \in \mathcal{A})\]

and $H$ a closed, convex and bounded subset of $E$.

**Theorem.** Let $f$ be a mapping from $H$ into $H$ such that:

(i) for all $\alpha \in \mathcal{A}$

\[|f(x_1) - f(x_2)|_\alpha \leq |x_1 - x_2|_\varphi \quad (x_1, x_2 \in H)\]

(ii) there is a compact set $M \subset H$ such that for every $x \in H$ the sequence $\{f^n(x)\}$ has an accumulation point in $M$ ($f^n = f \cdot f^{n-1}, f^1 = f$).

Then $f$ has a fixed point in $H$.

**Proof.** We can suppose that $H$ contains the null element of $E$ such that if we put $f_q = q \cdot f$ with $0 < q < 1, f_q(x) = (1 - q) \cdot 0 + qf(x) \in H$, that is $f_q$ is a contraction of $H$.

For every $\alpha \in \mathcal{A}$

\[|f_q(x_1) - f_q(x_2)|_\alpha \leq q |x_1 - x_2|_\varphi \quad (x_1, x_2 \in H)\]

such that, for $n = 1, 2, \ldots$ and $x \in H$

\[|f^{n+1}_q(x) - f^n_q(x)|_\alpha \leq q \cdot |f^n_q(x) - f^{n-1}_q(x)|_\varphi \leq \ldots \leq q^n |f_q(x) - x|_\varphi\]

It follows that

\[|f^{n+k}_q(x) - f^n_q(x)|_\alpha \leq (q^{n+k-1} + \ldots + q^n) |f_q(x) - x|_\varphi \leq \frac{q^n}{1 - q} |f_q(x) - x|_\varphi.\]

Consequently, for each $x \in H$ the sequence $\{f^n_q(x)\}$ is a Cauchy one. It results that there exists $x_q \in H$ such that:

\[|f(x_q) - x_q|_\varphi \leq 1 - q,\]
Indeed we can take \( x_q = f^n_2(x) \) with a sufficient large \( n \).

On the other hand

\[
|f(x_q) - x_q|_{q(\alpha)} \leq |f(x_q) - qf(x_q)|_{q(\alpha)} +
\]
\[
+ |f_q(x_q) - x_q|_{q(\alpha)} \leq (1 - q)|f(x_q)|_{q(\alpha)} + 1 =
\]
\[
= (1 - q)r
\]

where \( r \) is a positive number independent of \( x_q \) since the set \( H \) is bounded.

Hence, if \( n = 1, 2, \ldots \)

\[
|f^{n+1}(x_q) - f^n(x_q)|_\alpha \leq |f^n(x_q) - f^{n-1}(x_q)|_{q(\alpha)} \leq \ldots \leq (1 - q)r.
\]

Since the sequence \( \{f^n(x_q)\} \) has an accumulation point

\[
y_q \in M,
\]

for every \( \varepsilon > 0 \), there exists \( n \) such that:

\[
|f^n(x_q) - y_q|_\beta \leq \varepsilon \quad [\beta \geq \alpha, \ q(\chi)]
\]

From (4) and (5) it follows that:

\[
|f(y_q) - y_q|_\alpha \leq |f(y_q) - f^{n+1}(x_q)|_\alpha + |f^{n+1}(x_q) - f^n(x_q)|_\alpha +
\]
\[
+ |f^n(x_q) - y_q|_\alpha \leq |y_q - f^n(x_q)|_{q(\alpha)} + |f^{n+1}(x_q) - f^n(x_q)|_\alpha +
\]
\[
+ |f^n(x_q) - y_q|_\alpha \leq \varepsilon + (1 - q) . r + \varepsilon
\]

that is

\[
|f(y_q) - y_q|_\alpha \leq (1 - q) . r.
\]

Let \( \{q_i\} \) be a sequence of real numbers such that

\[
\lim_{i \to \infty} q_i = 1 \quad (0 < q_i < 1, \ i = 1, 2, \ldots).
\]

We consider a convergent subsequence \( \{y_{q'_i}\} \) of the corresponding sequence \( \{y_{q_i}\} \subset M. \)

If \( \lim_{i \to \infty} y_{q'_i} = y \in M \) it follows that

\[
\lim_{i \to \infty} |f(y_{q'_i}) - y_{q'_i}|_\alpha \leq \lim_{i \to \infty} (1 - q'_i) . r = 0.
\]

Since \( f \) is a continuous mapping, it holds

\[
|f(y) - y|_\alpha = 0
\]

for all \( \alpha \in \mathcal{A} \). Hence \( f(y) = y, \ q . e . d. \)

When the space \( E \) is normed there is obtained theorem 5 given by D. Göhde in [1].
REFERENCES


Faculty of Mathematics and Mechanics
Bucharest University