Oldřich Kowalski
A characterization of osculating maps

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In this paper we characterize osculating maps of higher order of a dif-
ferentiable map $V_n \rightarrow \tilde{V}_m$, $V_n$ being a simply connected manifold
and $\tilde{V}_m$ an affine space or a Lie group.

In the following, all manifolds, maps, vector bundles and their sections,
respectively are supposed to be differentiable of class $C^\infty$.

Let $V_n$ be a manifold of dimension $n$. For $p \in V_n$ let $f$ be a local
function at $p$ such that $f(p) = 0$. Then a $k$-jet $j^k_p(f)$ is called a covelocity
of order $k$ on $V_n$ at the point $p$. Let $T^{k*}(V_n)_p$ be a vector space of all
covelocities of order $k$ at $p$. Each linear form $X_p^{(k)}$ on $T^{k*}(V_n)_p$ is called
a vector of order $k$ at $p$. The set of all $X_p^{(k)}$ is a vector space $T_k(V_n)_p$.

We put $T_k(V_n) = \bigcup_{p \in V_n} T_k(V_n)_p$.

For any $k$, $T_k(V_n)$ is naturally a vector bundle over $V_n$ and $T_1(V_n) =$ $T(V_n)$ is the tangent bundle of $V_n$. (See [1], [3].)

Each vector $X_p^{(k)} \in T_k(V_n)$ is a linear differential operator on $V_n$
and, with respect to a local coordinate system $(u_1, ..., u_n)$ at $p$, it is
represented uniquely in the form

$$X_p^{(k)} = \sum_{i=1}^n a_i \frac{\partial}{\partial u^i} + \sum_{1 \leq i \leq j \leq n} a_{ij} \frac{\partial^2}{\partial u^i \partial u^j} + ... +$$

$$+ \sum_{1 \leq i_1 \leq ... \leq i_k \leq n} a_{i_1...i_k} \frac{\partial^k}{\partial u^{i_1}...\partial u^{i_k}}.$$

For any sequence of indices $0 \leq i_1 \leq ... \leq i_k \leq n$, $i_k > 0$, we can
introduce an operator $\frac{\partial^k}{\partial u^{i_1}...\partial u^{i_k}}$ putting inductively:

$$\frac{\partial^{l+1}}{\partial u^0 \partial u^{i_1}...\partial u^{i_l}} =$$

$$= \frac{\partial^l}{\partial u^{i_1}...\partial u^{i_l}} \text{ for each } l < k, 0 \leq j_1 \leq ... \leq j_l \leq n, j_l > 0.$$ Then (1) takes a simple form

$$(1') \quad X_p^{(k)} = \sum_{0 \leq i_1 \leq ... \leq i_k \leq n} a_{i_1...i_k} \frac{\partial^k}{\partial u^{i_1}...\partial u^{i_k}}.$$

In a coordinate neighbourhood $U \subset V_n$, the operators $\frac{\partial^k}{\partial u^{i_1}...\partial u^{i_k}}$, $0 \leq i_1 \leq ... \leq i_k \leq n$, $i_k > 0$, form a basis of $T_k(V_n)_q$ for each $q \in U$. 

A CHARACTERIZATION OF OSCULATING MAPS

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On the other hand, any vector \( X^{(k+1)}_p \in T_{k+1}(V_n) \) may be written in the form

\[
X^{(k+1)}_p = \sum_{i=1}^{r} X_{i, p} X^{(k)}_i,
\]

where \( X^{(k)}_i \) are suitable local sections of \( T_k(V_n) \) defined at \( p \), and \( X_{i, p} \in T(V_n)_p, i = 1, 2, \ldots, r \). For any \( l \leq k \) we have a canonical injection \( I_{l, k} : T_l(V_n) \to T_k(V_n) \).

Following P. Libermann, a symmetric surconnection \( S_k \) of order \( k \) on \( V_n \) is a bundle homomorphism \( S_k : T_{k+1}(V_n) \to T(V_n) \) such that \( S_k \circ I_{1, k+1} : T(V_n) \to T(V_n) = \) the identity map. (See [2].) It is easy to check that a symmetric surconnection \( S_1 : T_2(V_n) \to T(V_n) \) is an ordinary linear connection \( \nabla \) on \( V_n \) the torsion form of which vanishes. (See [3], p. 158.) Further, the successive interations \( \nabla^k \) of \( \nabla \) determine a sequence of symmetric surconnections \( S_k \) of orders \( k = 1, 2, \ldots \), if, and only if, the curvature form of \( \nabla \) vanishes, too. In this case, we can define the sequence \( \{S_k\} \) by induction: for \( X^{(k+1)}_p \in T_{k+1}(V_n), \ X^{(k+1)}_p = \sum_{i=1}^{r} X_{i, p} X^{(k)}_i \), we put

\[
S_k X^{(k+1)}_p = \sum_{i=1}^{r} \nabla_{X^{(k)}_i}(S_{k-1} X^{(k)}_i),
\]

It must be shown that (3) does not depend on a representation of \( X^{(k+1)}_p \) in the form (2). But this is just guaranteed by vanishing of both torsion and curvature forms of \( \nabla \). (The proof is routine and will be omitted.)

For \( l \leq k \), \( S_k \) is a prolongation of \( S_l \), i.e., \( S_l = S_k \circ I_{l+1, k+1} \) on \( T_{l+1}(V_n) \).

Note 1. If the curvature of \( \nabla \) is non-zero, the successive iterations \( \nabla^k \) define a sequence \( \{\tilde{S}_k\} \) of semiholonomic surconnections; see [2].

Note 2. On a paracompact \( V_n \), there are symmetric surconnections of any order \( k \). In fact, we can construct such a surconnection on each coordinate neighbourhood of \( V_n \) and then use a \( C^\infty \)-partition of unity subjected to a locally finite atlas of \( V_n \).

A differentiable map \( \varphi : V_n \to \tilde{V}_m \) induces canonically a sequence \( \{T_k(\varphi) : T_k(V_n) \to T_k(\tilde{V}_m)\} \) of bundle morphisms such that all the \( T_k(\varphi) \) diagrams \( T_k(V_n) \xrightarrow{\pi_k} T_k(\tilde{V}_m) \) are commutative, \( k = 1, 2, \ldots \). Let be \( \pi_k \downarrow \varphi \downarrow \pi_k \) \( V_n \xrightarrow{\varphi} \tilde{V}_m \) now \( \tilde{V}_m = A^m \), an affine space of dimension \( m \). Let us denote by \( W^m \) the associated vector space of \( A^m \) and for each \( x \in A^m \) let \( \omega_x : T(A^m)_x \to \)
\[ \rightarrow W^m \] be the canonical isomorphism. The maps \( \omega_x, x \in A^m \), determine a vector form \( \omega \) on \( A^m \), \( \omega : T(A^m) \rightarrow W^m \).

In \( A^m \), there is a canonical flat connection \( V \). Its successive iterations determine a canonical sequence \( \{S_k\} \) of symmetric surconnections on \( A^m \). In a linear coordinate system \( (x_1, \ldots, x_m) \) of \( A^m \), each \( S_k, k \geq 1 \), may be represented as follows: for \( X^{(k+1)}_p \in T^{(k+1)}(A^m) \), \( X^{(k+1)}_p = \sum a_{i_1 \ldots i_{k+1}} \frac{\partial}{\partial x^{i_1} \cdots \partial x^{i_{k+1}}} \), \( 0 \leq i_1 \leq \ldots \leq i_{k+1} \leq m \), \( i_{k+1} > 0 \) we have \( S_k(X^{(k+1)}_p) = \sum_{i=1}^{m} a_0 \ldots a_i \frac{\partial}{\partial x^i} \) is the first order part of \( X^{(k+1)}_p \). For \( k = 0 \) we put \( S_0: T(A^m) \rightarrow T(A^m) = \) the identity map. Let \( \varphi: V_n \rightarrow A^m \) be a smooth map. For any \( k \geq 1 \), we shall denote by \( \varphi^*_k: T_k(V_n) \rightarrow W^m \) the compositions of maps of the sequence

\[ \begin{align*}
T_k(V_n) &\xrightarrow{T_k(\varphi)} T_k(A^m) \xrightarrow{S_{k-1}} T(A^m) \xrightarrow{\omega} W^m.
\end{align*} \]

We can see that any \( \varphi^*_k \) is a composition of a bundle morphism \( \tilde{\varphi}_k: T_k(V_n) \rightarrow V_n \times W^m \) and a canonical projection \( pr_2: V_n \times W^m \rightarrow W^m \). In the regular case there is an index \( s \) such that \( \tilde{\varphi}_s \) is a bundle epimorphism. If \( (f_1, \ldots, f_m) \) is a basis of \( W^m \) corresponding to a linear coordinate system \( (x^1, \ldots, x^m) \), we have

\[ \varphi^*_k(X^{(k)}_p) = \sum_{i=1}^{m} [X^{(k)}_p(x^i \circ \varphi)] \cdot f_i. \]

For any \( l > k \), \( \varphi^*_l = \varphi^*_k \) holds on the bundle \( T_k(V_n) \) and hence it is possible to omit \( k \). From (5) we obtain immediately

\[ \varphi^*_k(X^{(l)}_p X^{(k)}_p) = X^{(k)}_p \varphi^*_k(X^{(k)}_p). \quad (k = 1, 2, \ldots) \]

(Here \( \varphi^*_k(X^{(l)}_p) \) is to be considered as a local vector function on \( V_n \) with values in \( W^m \).) Therefore, if \( \varphi^*_k(X^{(l)}_p) = \) const. for a local section \( X^{(k)}_p \) of the bundle \( T_k(V_n) \), we have \( \varphi^*_k(X^{(l)}_p X^{(k)}_p) = 0 \).

Our task is to prove the converse: in the regular case, the last property is characteristic for the maps \( \varphi^*_k \).

**Theorem 1.** Let \( V_n \) be a simply connected manifold and \( s \geq 1 \) an integer. Let be given a map \( \Phi: T^{(s+1)}(V_n) \rightarrow W^m \) of the form \( \Phi = pr_2 \circ \tilde{\Phi} \), where \( \tilde{\Phi}: T^{(s+1)}(V_n) \rightarrow V_n \times W^m \) is a bundle morphism and \( pr_2: V_n \times W^m \rightarrow W^m \) is a canonical projection. Suppose that

a) the restriction of \( \tilde{\Phi} \) to the subbundle \( T_s(V_n) \) is a bundle epimorphism,

b) if \( X_p \in T(V_n) \) and \( X^{(s)}_p \) is a local section of \( T_s(V_n) \) defined at \( p \) such that \( \Phi(X^{(s)}_p) = \) const., then \( \Phi(X_p X^{(s)}_p) = 0 \). Under these assumptions
there is exactly one map \( \varphi: V_n \rightarrow A^n \) satisfying initial condition \( \varphi(p) = x \) and such that \( \varphi^* = \Phi \) on \( T(V_n) \). Moreover, we have \( \varphi^* = \Phi \) on the whole bundle \( T_{s+1}(V_n) \).

Proof. Let be given \( p \in V_n \) and a basis \((f_1, \ldots, f_m)\) of \( W^m \). Denote by \( \nu \) the dimension of a fibre of \( T_s(V_n) \). As \( \Phi \) induces a bundle epimorphism \( T_s(V_n) \rightarrow V_n \times W^m \), the following assertion may be easily verified: there is a coordinate neighbourhood \( U(u_1, \ldots, u_m) \) at \( p \) and local sections \( X^{(v)}_1, \ldots, X^{(v)}_{\nu} \) of \( T_s(V_n) \) over \( U \) such that (i) the vectors \( X^{(v)}_1, \ldots, X^{(v)}_{\nu} \) are linearly independent, (ii) we have

\[
\Phi(X^{(v)}_i) = f_i, \quad i = 1, 2, \ldots, m
\]

\[
\Phi(X^{(v)}_i) = 0, \quad i = m + 1, \ldots, \nu
\]

identically on \( U \).

Put

\[
X^{(v)}_i = \sum_{0 \leq t_1 \leq \ldots \leq t_\nu \leq m} a^{t_1 \ldots t_\nu}_{i(p)} \left( \frac{\partial^s}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right), \quad i = 1, \ldots, \nu,
\]

then the determinant \( |a^{i}^{t_{1} \ldots t_{\nu}}_{(p)}| \neq 0 \). Now

\[
\frac{\partial}{\partial u^k} X^{(v)}_i = \sum \left\{ \frac{\partial a^{t_1 \ldots t_\nu}_{i(p)}}{\partial u^k} \left( \frac{\partial^s}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right) + a^{t_1 \ldots t_\nu}_{i(p)} \left( \frac{\partial^{s+1}}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right) \right\},
\]

\[
\frac{\partial}{\partial u^k} \Phi(X^{(v)}_i) = \sum \left\{ \frac{\partial a^{t_1 \ldots t_\nu}_{i(p)}}{\partial u^k} \Phi \left( \frac{\partial^s}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right) + a^{t_1 \ldots t_\nu}_{i(p)} \frac{\partial}{\partial u^k} \Phi \left( \frac{\partial^{s+1}}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right) \right\} = 0,
\]

and according to the assumption \( b \) of the Theorem,

\[
\Phi \left( \frac{\partial}{\partial u^k} X^{(v)}_i \right) = \sum \left\{ \frac{\partial a^{t_1 \ldots t_\nu}_{i(p)}}{\partial u^k} \Phi \left( \frac{\partial^s}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right) + a^{t_1 \ldots t_\nu}_{i(p)} \Phi \left( \frac{\partial^{s+1}}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right) \right\} = 0.
\]

Thus we have, for any \( k = 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, \nu \),

\[
\sum a^{t_1 \ldots t_\nu}_{i(p)} \left\{ \Phi \left( \frac{\partial^{s+1}}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right)^{(p)} \Phi \left( \frac{\partial^s}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right)^{(p)} \right\} = 0.
\]

In view of \( |a^{i_{1} \ldots i_{\nu}}_{(p)}| \neq 0 \),

\[
\Phi \left( \frac{\partial^{s+1}}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right)^{(p)} = \frac{\partial}{\partial u^k} \Phi \left( \frac{\partial^s}{\partial u^{t_1} \ldots \partial u^{t_\nu}} \right)^{(p)}
\]
for any sequence $0 \leq i_1 \leq i_2 \leq \ldots \leq k \leq \ldots \leq i_s \leq n$, $k > 0$. Hence we obtain easily

$$\Phi(X_pX^{(s)}) = X_p\Phi(X^{(s)})$$

for any vector $X_p \in T(V_n)$ and any local section $X^{(s)}$ of $T_s(V_n)$ defined at $p$.

To complete our proof we shall use the Frobenius Theorem. Put $D = V_n \times \mathbb{A}^m$. At each point $\alpha \in D$, $\alpha = (p, x)$ we have $T(D)_\alpha = T(V_n)_p + T(\mathbb{A}^m)_x$. We shall construct on $D$ a differentiable distribution $\Delta_\alpha$ of dimension $n$ as follows: for any $\alpha \in D$, $\alpha = (p, x)$, let $\Delta_\alpha \subset T(D)_\alpha$ be a linear subspace of all vectors of the form $X_p + \omega^{-1}_x \Phi(X_p)$, $X_p \in T(V_n)_p$. The distribution $\Delta_\alpha$ is involutive. In fact, let $\pi$: $D \rightarrow \bar{V}_n$ be a canonical projection. For any $\alpha = (p, x) \in D$, there are linearly independent vector fields $X_1, \ldots, X_n$ defined on a neighbourhood $U \ni p$. Then the vector fields $\tilde{X}_i, \tilde{X}_j$ do not depend essentially on $y$, we have $\omega^{-1}_x \Phi(X_{i,p}) \tilde{X}_i = 0$, $\omega^{-1}_x \Phi(X_{j,p}) \tilde{X}_j = 0$, and hence $[\tilde{X}_i, \tilde{X}_j]_\alpha = [X_i, X_j]_p + X_j, \omega^{-1}_x \Phi(X_{i,q}) - X_i, \omega^{-1}_x \Phi(X_{j,p}) = [X_i, X_j]_p + \omega^{-1}_x \{X_i, \Phi(X_j) - X_j, \Phi(X_i)\} = [X_i, X_j]_p + \omega^{-1}_x \Phi([X_i, X_j]_p)$, according to (7).

There is only one maximal integral manifold $\tilde{V}_n$ of the distribution $\Delta_\alpha$, passing through a prescribed point $\alpha_0 \in D$. Then for any $\alpha \in \tilde{V}_n$, $\alpha = (p, x)$, we have $\det [T(\tilde{V}_n)] = \det (\Delta_\alpha) = T(V_n)_p$. Hence $\pi$ is a local diffeomorphism. Since $\Delta$ is invariant with respect to all transformations of $D$ of the form $(q, y) \rightarrow (q, y + \alpha)$, $\tilde{V}_n$ is a covering space of $V_n$. As $V_n$ is simply connected, $\pi$ is a diffeomorphism. If $\varphi$: $D \rightarrow \mathbb{A}^m$ is a canonical projection, we obtain a map $\varphi$: $V_n \rightarrow \mathbb{A}^m$, $\varphi = \varphi \circ \pi^{-1}$. Here $\det (X_p) = \omega^{-1}_x \Phi(X_p)$ for any $X_p \in T(V_n)_p$ and consequently, in view of (4), $\Phi = \omega \circ \varphi$ on $T(V_n)$. Finally, from (6), (7), we see, step by step, that $\varphi^* = \Phi$ on $T_2(V_n)$, $T_3(V_n)$, ..., $T_{s+1}(V_n)$, q.e.d.

As an application of Theorem 1, we can re-prove a result of Kočandře (see [6]). First we shall present some concepts of [6]. Let be given a covariant tensor $t(x_1, \ldots, x_r)$ of degree $r$ on $\mathbb{W}^m$. We shall denote by $\hat{i}S$ the set of all vectors $y \in \mathbb{W}^m$ such that $t(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_r) = 0$ for arbitrary vectors $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r \in \mathbb{W}^m$. The intersection $S = \bigcap_{i=1}^n \hat{i}S$ is called the singular space of $t$. The automorphism group of the tensor $t$ is the group of all transformations $g \in GL(m)$ such that $t(x_1, \ldots, x_r) = t(x_1g, \ldots, x rg)$ for each $x_1, \ldots, x_r \in \mathbb{W}^m$. It is a Lie
subgroup $G^0 \subset GL(m)$). Let be given a fixed regular tensor $t (x_1, \ldots, x_r)$ on $W^m$, i.e., such that its singular space $S = \{0\}$. If $\varphi: V_n \to A_m$ is a map, $\varphi^*: T_k(V_n) \to W^m$ the induced maps given by (4), $k = 1, 2, \ldots,$ we can define an $r$-linear form $t_k = t \circ (\otimes \varphi^*)$ on $T^*_1(V_n)$ for each $k = 1, 2, \ldots$. For each $k > l$, $t_k^*$ coincides with $t_l^*$ on $T^*_1(V_n)$. The sequence $\{t_k^*\}$ of multilinear forms is called the fundamental tensor of the manifold $V_n$. Now, the main result of [6] is a characterization of the fundamental tensor.

Let us consider the following conditions:

I. On $V_n$, there is given a differentiable tensor $t^*$, covariant of degree $r$, acting at each point $p \in V_n$ on the $(k_0 + 1)$-vectors from $T_{k_0 + 1}(V_n)_p$, $k_0$ is a given number.

Let us denote by $S_{k_0, p}, S_{k_0 + 1, p}$ the singular spaces of $t^*$ on $T_{k_0}(V_n)_p$ and $T_{k_0 + 1}(V_n)_p$, respectively.

II. For any differentiable fields of $k_0$-vectors $X_1^{(k_0)}, \ldots, X_r^{(k_0)}$ and any vector $Y_p \in T(V_n)_p$, we have $Y_p t^*(X_1^{(k_0)} \ldots, X_r^{(k_0)}) = \sum_{i=1}^r t^*(X_1^{(k_0)} \ldots, X_i^{-1, p}, Y_p X_i^{(k_0)} \ldots, X_r^{(k_0)})$.

III. $\dim T_{k_0}(V_n)_p/S_{k_0, p} = \dim T_{k_0 + 1}(V_n)_p/S_{k_0 + 1, p} = m$ for each point $p \in V_n$; $S_{k_0 + 1, p} \cap T(V_n)_p = \{0\}$ for each $p \in V_n$.

Let $P$ denote the principal fibre bundle of all bases of the spaces $T_{k_0}(V_n)_p/S_{k_0, p}, p \in V_n$.

IV. To each point $p \in V_n$ there is a neighbourhood $U \subset V_n$ of $p$ and a local section $s$ of the fibre bundle $P$ over $U$ such that the components of $t^*$ with respect to the basis $s_q$ are constant functions of $q$ on $U$.

V. There is a point $p \in V_n$ such that the vector space $W^m$ with the given tensor $t$ is isomorphic to the space $T_{k_0}(V_n)_p/S_{k_0, p}$ with the tensor $t^*$.

Let us introduce the abbreviations $T_{k_0} = T_{k_0}(V_n), S_{k_0} = \bigcup_{q \in V_n} S_{k_0, q}$, and similarly for the index $k_0 + 1$. From III we obtain easily a commutative diagram $S_{k_0} \longrightarrow S_{k_0 + 1}$ over $V_n$, and a canonical isomorphism $\sigma: T_{k_0}/S_{k_0} \to T_{k_0 + 1}/S_{k_0 + 1}$ of factor bundles. Let

$\pi_{k_0}: T_{k_0} \to T_{k_0}/S_{k_0}, \pi_{k_0 + 1}: T_{k_0 + 1} \to T_{k_0 + 1}/S_{k_0 + 1}$

be canonical projections. We have a commutative diagram

\[
\begin{array}{ccc}
T_{k_0} & \longrightarrow & T_{k_0 + 1} \\
\downarrow \pi_{k_0} & & \downarrow \pi_{k_0 + 1} \\
T_{k_0}/S_{k_0} & \longrightarrow & T_{k_0 + 1}/S_{k_0 + 1} \\
I_{k_0, k_0 + 1} & \longrightarrow & T_{k_0 + 1}/S_{k_0 + 1} \\
\end{array}
\]
Let $G_0$ be the automorphism group of the tensor $t$ on $W^m$. We can prove that $V$ is satisfied at each point $q \in V_n$. Let $P^0_q$ be the set of all isomorphisms $\chi_q: T_{k_0}(q)/S_{k_0} \to W^m$ with the property $V$, i.e., such that $t^* = t \circ \otimes (\chi_q \circ \pi_{k_0}, q)$. Then $P_0 = \bigcup_{q \in V_n} P^0_q$ is a principal fibre bundle over $V_n$ with the structural group $G_0$. If we choose a fixed basis $Q^\nu$ of $W^m$, we obtain a canonical injection $P^0 \to P$. Let be given $\phi(x, q)$ be the set of all isomorphisms $X^*_q: T_k^*(q) \to W^m$ such that $t^* = t \circ \otimes (\chi_q \circ \pi_{k_0}, q)$. Then $P^0 = \bigcup P^0_q$ is a principal fibre bundle over $V_n$ with the structural group $G_0$.

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The conditions of Theorem 1 are satisfied and consequently, there is a map $\varphi: V_n \to A^m$ such that $\Phi = \varphi_{k+1}^*$ on $T_{k+1}(V_n)$. Since the restriction of $\Phi_q$ to $T(V_n)_q$ is a monomorphism (the second condition of III), we can see easily that $\varphi$ is an immersion. Now from the construction of the principal bundle $P^0$ we see that, on each $T_{k+1}(V_n)_q$,

$$t_q^* = t \circ (\chi \circ h_q \circ \pi_{k+1,q}) = t \circ (\otimes \Phi_q) = t \circ (\otimes \varphi_{k+1,q}),$$

which proves our assertion for $V_n$ simply connected.

In case $V_n$ to be not simply connected, let us consider the universal covering manifold $\tilde{V}_n$ of $V_n$ (see [8]). Then the proof will be easily traced back to the preceding case.

In the second part of this Paper we shall try to generalize Theorem 1, at least in a weakened form, to the case when $A^m$ is replaced by an arbitrary Lie group. So, let $G$ be a Lie group, $g$ its Lie algebra. For $X_g \in T(G)$ let us denote by $\omega(X_g)$ the left invariant vector field on $G$ determined by $X_g$. Then $\omega: T(G) \to g$ is a vector form on $G$, each partial map $\omega_g: T(G)_g \to g$ being an isomorphism. Let $S_k: T_{k+1}(G) \to T(G)$ be a surconnection on $G$ and $\varphi: V_n \to G$ a differentiable map. Then we have a sequence of maps, analogous to (4):

$$(8) \quad T_{k+1}(V_n) \xrightarrow{T_{k+1}(\varphi)} T_{k+1}(G) \xrightarrow{S_k} T(G) \xrightarrow{\omega} g.$$

Let $\varphi^*: T_{k+1}(V_n) \to g$ denote the composed map of the sequence.

Obviously $\varphi^*$ may be written as a composition $T_{k+1}(V_n) \to V_n \times g \to g$, of a bundle morphism and a canonical projection.

**Proposition 1.** There is a map $\Psi^*: T(V_n) \otimes T(V_n) \to g$, a composition of a bundle morphism $T(V_n) \otimes T(V_n) \to V_n \times g$ and the canonical projection $\text{pr}_2: V_n \times g \to g$, with the following property:

$$(9) \quad \varphi^*(X_pX^{(k)}) = X_p\varphi^*(X^{(k)}) + \Psi^*(X_p \otimes X^{(k)}),$$

for any vector $X_p \in T(V_n)$ and any local section $X^{(k)}$ of $T(V_n)$ defined at $p$.

**Proof.** Let be given $X_p \in T(V_n)_p$, $X^{(k)}_p \in T(V_n)_p$. Let $X^{(k)}$ be a local section of $T_{k}(V_n)$ passing through $X^{(k)}_p$. It suffices to prove that the expression $\varphi^*(X_pX^{(k)}) - X_p\varphi^*(X^{(k)})$ depends on $X_p$, $X^{(k)}_p$ only and that it is linear in each argument. Choose a local coordinate system $(u_1, \ldots, u_n)$ at $p$ and put

$$X_p = \sum_{i=1}^{n} a^i \frac{\partial}{\partial u^i}, \quad X^{(k)} = \sum_{0 \leq i_1 \leq \ldots \leq i_k \leq n \atop i_k > 0} a^{i_1, \ldots, i_k}(q) \frac{\partial^k}{\partial u^{i_1} \ldots \partial u^{i_k}}.$$
Then

\[ X_p X^{(k)} = \sum \left\{ a^i \frac{\partial a_{i_1} \ldots a_{i_k}}{\partial u^i} \cdot \frac{\partial}{\partial u^{i_1} \ldots \partial u^{i_k}} + a^i a_{i_1} \ldots a_{i_k}(p) \frac{\partial^{i_1+1}}{\partial u^{i_1} \ldots \partial u^i \ldots \partial u^{i_k}} \right\} \]

\[ \varphi^*(X_p X^{(k)}) - X_p \varphi^*(X^{(k)}) = \sum a^i a_{i_1} \ldots a_{i_k}(p) \left\{ \varphi_p^* \left( \frac{\partial^{i_1+1}}{\partial u^{i_1} \ldots \partial u^i \ldots \partial u^{i_k}} \right) - \left( \frac{\partial}{\partial u^i} \right) \varphi_p^* \left( \frac{\partial}{\partial u^{i_1} \ldots \partial u^{i_k}} \right) \right\} . \]

This proves our assertion.

**Proposition 2.** For any \( X_p, Y_p \in T(V_n) \), we have \( \Psi^*(X_p \otimes Y_p) - Y_p \otimes X_p = [\varphi^*(X_p), \varphi^*(Y_p)] \), where \( \Psi \) is the bracket operation in the algebra \( \mathfrak{g} \).

**Proof.** Let us remind the equations \( d\omega = -1/2 [\omega, \omega] \), \( d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \), where \( \omega: T(G) \rightarrow \mathfrak{g} \) is the canonical form. (See [5], for instance). These equations are still valid if we substitute the form \( \omega \) by the form \( \omega' = \omega \circ \varphi \), \( d\varphi: T(V_n) \rightarrow T(G) \) being the tangent map of \( \varphi \). According to (8), we have \( \omega' = \varphi^* \) on \( T(V_n) \). Let \( X, Y \) be local tangent fields at \( p \) passing through \( X_p, Y_p \), respectively. Then

\[ \varphi^*(X_p Y) - \varphi^*(Y_p X) = \varphi^*([X, Y])_p = X_p \omega'(Y) - Y_p \omega'(X) - d\omega'(X_p, Y_p) = X_p \omega'(Y) - Y_p \omega'(X) + [\omega'(X_p), \omega'(Y_p)] = X_p \varphi^*(Y) - Y_p \varphi^*(X) + [\varphi^*(X_p), \varphi^*(Y_p)] . \]

On the other hand

\[ \varphi^*(X_p Y) - \varphi^*(Y_p X) = X_p \varphi^*(Y) + \Psi^*(X_p \otimes Y_p) - Y_p \varphi^*(X) - \Psi^*(Y_p \otimes X_p) . \]

This proves our assertion.

**Theorem 2.** Let \( V_n \) be a simply connected manifold, \( G \) a Lie group with the algebra \( \mathfrak{g} \). Let be given differentiable maps \( \Phi: T_{s+1}(V_n) \rightarrow \mathfrak{g}; \psi: T(V_n) \otimes T_s(V_n) \rightarrow \mathfrak{g}, \Phi: T_{s+1}(V_n) \rightarrow V_n \times \mathfrak{g}, \tilde{\psi}: T(V_n) \otimes T_s(V_n) \rightarrow V_n \times \mathfrak{g} \), respectively and of the canonical projection \( pr_2: V_n \times \mathfrak{g} \rightarrow \mathfrak{g} \). Suppose that

a) the restriction of \( \Phi \) to the subbundle \( T_s(V_n) \) is a bundle epimorphism,

b) if \( X_p \in T(V_n) \) and \( X^{(i)} \) is a local section of \( T_s(V_n) \) defined at \( p \) such that \( \Phi(X^{(i)}) = \text{const.} \), then \( \Phi(X_p X^{(i)}) = \psi(X_p \otimes X^{(i)}) \),

c) for any two vectors \( X_p, Y_p \in T(V_n) \) we have

\[ \psi(X_p \otimes Y_p - Y_p \otimes X_p) = [\Phi(X_p), \Phi(Y_p)] . \]
Then there is exactly one map \( \varphi: V_n \to G \) satisfying initial condition \( \varphi(p) = g \) and such that \( \Phi = \omega \circ d\varphi \) on \( T(V_n) \).

Proof. An argument like that in the proof of Theorem 1 shows that

\begin{equation}
\Phi(X_pX^{(\omega)}) = X_p \Phi(X^{(\omega)}) + \psi(X_p \otimes X^{(\omega)}_p)
\end{equation}

for any vector \( X_p \in T(V_n) \) and any local section \( X^{(\omega)} \) of \( T_s(V_n) \) defined at \( p \). Let us define a distribution \( \Delta_\alpha \) on \( D = V_n \times G \) as follows: if \( \alpha = (p, g) \in D \), then \( \Delta_\alpha \) consists of all vectors of the form \( X_p + \omega_g^{-1}\Phi(X_p) \), \( X_p \in T(V_n)_p \). For two vector fields \( \bar{X}_\beta = X_q + \omega_h^{-1}\Phi(X_q) \), \( \bar{Y}_\beta = Y_q + \omega_h^{-1}\Phi(Y_q) \) belonging to \( \Delta \) in a neighbourhood of \( \alpha = (p, g) \) we obtain \([\bar{X}, \bar{Y}]_\alpha = [X, Y]_p + \omega_g^{-1}X_p \Phi(Y) - \omega_g^{-1}Y_p \Phi(X) + \omega_g^{-1}[\Phi(X_p), \Phi(Y_p)] = [X, Y]_p + \omega_g^{-1}\Phi([X, Y]_p)\). It is a consequence of (10) and of the assumption c) of the Theorem. Thus the distribution \( \Delta \) is involutive. The rest of the proof is the same as at Theorem 1.

Note 3. If the restriction of \( \Phi \) to the subbundle \( T(V_n) \) is a monomorphism, then \( \varphi \) is an immersion.

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There is an interesting special case when we may prove a stronger result, an exact analog of Theorem 1. It is the case when \( [[X, Y], Z] = 0 \) for any \( X, Y, Z \in g \). As known, on any Lie group \( G \) there is exactly one connection \( \nabla \) with the following properties:

a) The geodesics of \( \nabla \) are the integral curves of left invariant vector fields on \( G \),

b) the torsion form \( T(X, Y) = 0 \). (See [5], Chapter 6.) We have

\begin{equation}
\nabla_X Y = \omega_g^{-1}\{X_g\omega(Y) + \frac{1}{2}[\omega(X_g), \omega(Y_g)]\}
\end{equation}

for any vector \( X_g \in T(G)_g \) and any vector field \( Y \) at \( g \). Finally, the curvature form is given by \( R(X, Y, Z) = 1/4[[X, Y], Z] \). In our special case we have \( R(X, Y, Z) = 0 \) and consequently, the iterations \( \nabla_k \) generate a canonical sequence \( \{S_k\} \) of symmetric surconnections on \( G \). According to (3), (11) we have

\begin{equation}
S_k(X_gX^{(\omega)}) = \nabla_XS_{k-1}(X^{(\omega)}) = \omega_g^{-1}\{X_g[\omega \circ S_{k-1}] (X^{(\omega)}) + \frac{1}{2} [\omega(X_g), [\omega \circ S_{k-1}] (X^{(\omega)})]\}
\end{equation}

for \( k \geq 2 \).
Now, for any differentiable map \( \varphi: V_n \to G \) and any \( k \geq 0 \) we can define a map \( \varphi^*_{k+1}: T_{k+1}(V_n) \to g \) by \( \varphi^*_{k+1} = \omega \circ S_k \circ T_{k+1}(\varphi) \). (Here \( \varphi^*_1 = \omega \circ T_1(\varphi) \).) From (11), (12) we get a formula

\[
\varphi^*(X_pX^{(k)}) = X_p \varphi^*(X^{(k)}) + \frac{1}{2} \left[ \varphi^*(X_p), \varphi^*(X^{(k)}) \right].
\]

If the mapping \( \psi: T(V_n) \otimes T_s(V_n) \to g \) introduced in Theorem 2 is given by \( \psi(X_p \otimes X^{(m)}_p) = 1/2[\Phi(X_p), \Phi(X^{(m)}_p)] \), the condition \( c \) is fulfilled. From Theorem 2 and (10), (13), we obtain the following result: there is exactly one map \( \varphi: V_n \to G \) satisfying an initial condition and such that \( \varphi^* = \Phi \) on \( T(V_n) \). Moreover, we have \( \varphi^* = \Phi \) on the whole \( T_{s+1}(V_n) \).

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