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ON CARTESIAN PRODUCTS

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Dedicated to Professor OTAKAR BORŮVKA to the 70th anniversary of his birthday

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INTRODUCTION

In the present paper the following problem is solved: Let $f: \bigtimes_{k\in K} U_k \to S$, $f': \bigtimes_{k'\in K'} U'_{k'} \to S$ be bijections. We study the existence of sets $U_{kk'}$ $((k, k') \in K \times K')$ and of bijections $f'_k: \bigtimes_{k'\in K'} U_{kk'} \to U_k$, $f_{k'}: \bigtimes_{k\in K} U_{kk'} \to U'_{k'}$ $(k \in K, k' \in K')$ with the property $f(f'_k(u_{kk'})_{k'\in K'})_{k\in K} = f'(f_{k'}(u_{kk'})_{k\in K})_{k'\in K'}$ for every system of elements $u_{kk'} \in U_{kk'}$ $((k, k') \in K \times K')$. Necessary and sufficient conditions for the existence of such sets $U_{kk'}$ and bijections f'_k, $f_{k'}$ can be found in the Main Theorem of the paper.

The Main Theorem contains the set theoretical kernel of Reimer's investigations [1] and represents a basis for the proof of theorems about the existence of a common refinement of two direct decompositions of a relational system (see e.g. [2]). This application will be shown in a latter paper.

In his books [3], [4], Borůvka tried to build up the group theory step by step: In the first part of the book there are purely set theoretical studies concerning mappings and decompositions of sets. By adding a binary operation we get the theory of homomorphisms and quotients on groupoids from these set theoretical results. Our theory of Cartesian products gives a similar possibility to build up the theory of direct products of algebraic structures starting with purely set theoretical concepts and theorems concerning Cartesian products.

It seemed to be natural and convenient to substitute the concept of a Cartesian product by an algebraic structure and to study, instead of Cartesian products, the so called Cartesian algebras.

1. ADMISSIBLE QUADRUPLES AND CARTESIAN ALGEBRAS

1.1. Definition. Let S, K be non-empty sets, U_k a set for every $k \in K$, f a bijection of $\bigtimes_{k\in K} U_k$ onto S, $n \in S$ an arbitrary element. Then the quadruple $(S, (U_k)_{k\in K}, f, n)$ is called an *admissible quadruple*.

1.2. Definition. Let $(S, (U_k)_{k\in K}, f, n)$ be an admissible quadruple. We put $(n_k)_{k\in K} = f^{-1} n$.

For every $(u_k)_{k\in K} \in \bigtimes_{k\in K} U_k$ and every $k_0 \in K$ we put $p_{k_0}(u_k)_{k\in K} = u_{k_0}$.

For every $k_0 \in K$ and every $\bar{u}_{k_0} \in U_{k_0}$ we put $o_{k_0}\bar{u}_{k_0} = (u_k)_{k\in K} \in \bigtimes_{k\in K} U_k$ where $u_k = n_k$ for every $k \in K$, $k \neq k_0$, $u_{k_0} = \bar{u}_{k_0}$.

We put $q_k = f o_k p_k f^{-1}$ for every $k \in K$.

We put $g(s_k)_{k\in K} = f(p_k f^{-1} s_k)_{k\in K}$ for every $(s_k)_{k\in K} \in \bigtimes_{k\in K} q_k S$.

1.3. Lemma. *Let $(S, (U_k)_{k\in K}, f, n)$ be an admissible quadruple. Then the following assertions hold true:*

(*i*) $p_k o_k = \mathrm{id}_{U_k}$ *for every* $k \in K$.

(*ii*) *If* $k, l \in K$ *and* $(u_m)_{m\in K} \in \bigtimes_{m\in K} U_m$ *then*

$$o_k p_k o_l p_l (u_m)_{m\in K} = \begin{cases} o_k u_k & \text{if } l = k \\ (n_m)_{m\in K} & \text{if } l \neq k. \end{cases}$$

(*iii*) $(u_l)_{l\in K} = (p_k(u_l)_{l\in K})_{k\in K}$ *for every* $(u_l)_{l\in K} \in \bigtimes_{l\in K} U_l$.

The proof is very simple.

1.4. Definition. Let K be a non-empty set, $(S, n, (q_k)_{k\in K}, g)$ a partial algebra (see [5]) where n is a complete nullary, q_k a complete unary operation for every $k \in K$, and g a partial operation of type K. Let the following axioms hold:

(*a*) $$q_k q_l = \begin{cases} q_k \text{ for } k, l \in K,\ k = l. \\ n \text{ for } k, l \in K,\ k \neq l. \end{cases}$$

(*b*) $$\mathrm{dom}\ g = \bigtimes_{k\in K} q_k S.$$

(*c*) $$g(q_k s)_{k\in K} = s \text{ for every } s \in S.$$

(*d*) $q_k g(s_l)_{l\in K} = s_k$ for every $k \in K$ and every $(s_l)_{l\in K} \in \bigtimes_{l\in K} q_l S$.

Then the algebra $(S, n, (q_k)_{k\in K}, g)$ is called a *Cartesian algebra* (abbreviation: *C-algebra*).

1.5. Lemma. *Let $(S, n, (q_k)_{k\in K}, g)$ be a C-algebra. Then the following assertions hold true:*

(*i*) $$q_k s = \begin{cases} s \text{ for every } k \in K,\ s \in q_k S. \\ n \text{ for every } k, l \in K,\ k \neq l,\ s \in q_l S. \end{cases}$$

(*ii*) $q_k n = n$ *for every* $k \in K$.

(*iii*) *If* $k \in K$, $s_k \in q_k S$, $s_l = n$ *for every* $l \in K$, $l \neq k$ *then* $g(s_l)_{l\in K} = s_k$.

Proof. (*i*) Let $s \in q_l S$; then there exists an element $t \in S$ such that $s = q_l t$. If $k = l$ then $q_k s = q_k q_k t = q_k t = s$, if $k \neq l$ then $q_k s = q_k q_l t = n$ according to the axiom (*a*).

(*ii*) If $|K| = 1$ then $g q_k = \mathrm{id}_S$ according to the axiom (*c*). Thus, $q_k n = \mathrm{id}_S q_k n = g q_k q_k n = g q_k n = \mathrm{id}_S n = n$ according to the axiom (*a*). If $|K| > 1$ then we take a pair of indices $l, m \in K$, $l \neq m$. Then $q_l q_m s = n$ for an arbitrary $s \in S$ according to the axiom (*a*). Thus, $q_k n = q_k(q_l q_m s)$. If $k \neq l$ then the last element is $q_k q_l(q_m s) = n$, if $k = l$ then the last element is $q_k q_k(q_m s) = q_k q_m s = q_l q_m s = n$ according to the axiom (*a*).

(*iii*) If $s_k \in q_k S$ then

$$q_l s_k = \begin{cases} n \text{ for every } l \in K,\ l \neq k, \\ s_k \text{ for } l \in K,\ l = \mathrm{k} \end{cases}$$

according to (*i*). Thus, $q_l s_k = s_l$ for every $l \in K$. Therefore $s_k = g(q_l s_k)_{l \in K} = g(s_l)_{l \in K}$ according to the axiom (*c*).

1.6. Definition. Let $(S, (U_k)_{k \in K}, f, n)$ be an admissible quadruple. We put $\boldsymbol{A}(S, (U_k)_{k \in K}, f, n) = (S, n, (q_k)_{k \in K}, g)$ where the operations n, q_k for every $k \in K$ and g are defined according to 1.2.

Then $\boldsymbol{A}(S, (U_k)_{k \in K}, f, n)$ is clearly a partial algebra which is similar to a C-algebra.

1.7. Lemma. *Let $(S, (U_k)_{k \in K}, f, n)$ be an admissible quadruple. Then $\boldsymbol{A}(S, (U_k)_{k \in K}, f, n)$ is a C-algebra.*

Proof. 1. For every $k, l \in K$ we have $q_k q_l = f o_k p_k f^{-1} f o_l p_l f^{-1} = f o_k p_k o_l p_l f^{-1}$. The last mapping is $f o_k p_k f^{-1} = q_k$ if $k = l$ and $f(n_m)_{m \in K} = n$ if $k \neq l$ according to 1.3 (*i*) and (*ii*). Thus the axiom (*a*) holds true.

2. Clearly, $\mathrm{dom}\, g = \bigtimes_{k \in K} q_k S$ which is the axiom (*b*).

3. Let $s \in S$ be an arbitrary element. We have $g(q_k s)_{k \in K} = f(p_k f^{-1} f o_k p_k f^{-1} s)_{k \in K} = f(p_k o_k p_k f^{-1} s)_{k \in K} = f(p_k f^{-1} s)_{k \in K} = f f^{-1} s = s$ according to 1.3 (*i*) and (*iii*). Thus, the axiom (*c*) is proved.

4. Let us have $k \in K$, $(s_l)_{l \in K} \in \bigtimes_{l \in K} q_l S$. Thus, $q_k g(s_l)_{l \in K} = f o_k p_k f^{-1} f(p_l f^{-1} s_l)_{l \in K} = f o_k p_k (p_l f^{-1} s_l)_{l \in K} = f o_k p_k f^{-1} s_k = q_k s_k$. As $s_k \in q_k S$ then there exists an element $t \in S$ with the property $s_k = q_k t$. Thus $q_k s_k = q_k q_k t = q_k t = s_k$ according to the part 1 of this proof. Thus, the axiom (*d*) is proved.

1.8. Remark. If $(S, (U_k)_{k \in K}, f, n)$ is an admissible quadruple and $(S, n, (q_k)_{k \in K}, g) = \boldsymbol{A}(S, (U_k)_{k \in K}, f, n)$ then $q_k S = f o_k p_k f^{-1} S = f o_k U_k$ for every $k \in K$.

1.9. Lemma. *Let $(S, n, (q_k)_{k \in K}, g)$ be a C-algebra. Then $(S, (q_k S)_{k \in K}, g, n)$ is an admissible quadruple and $\boldsymbol{A}(S, (q_k S)_{k \in K}, g, n) = (S, n, (q_k)_{k \in K}, g)$.*

Proof. It follows from the axiom (*b*) that g is a mapping of $\bigtimes_{k\in K} q_kS$ into S, from the axiom (*c*) that g is surjective and from the axiom (*d*) that g is injective. Thus $(S, (q_kS)_{k\in K}, g, n)$ is an admissible quadruple.

We have to prove that the unary operations and the partial operation of $\boldsymbol{A}(S, (q_kS)_{k\in K}, g, n)$ coincide with q_k and g respectively.

It follows from 1.5 (*ii*) and (*iii*): If $t_k = n$ for every $k \in K$ then $g(t_k)_{k\in K} = n$. Thus, $g^{-1}n = (t_k)_{k\in K}$ where $t_k = n$ for every $k \in K$. Thus, $n_k = n$ for every $k \in K$.

It follows that for every $k \in K$, $\bar{u}_k \in q_kS$, we have $o_k\bar{u}_k = (u_l)_{l\in K}$ where $u_k = \bar{u}_k$ and $u_l = n$ for $l \in K$, $l \neq k$. As $q_l\bar{u}_k = \bar{u}_k$ for $l \in K$, $l = k$ and $q_l\bar{u}_k = n$ for $l \in K$, $l \neq k$ according to 1.5 (*i*) we get

$$o_k\bar{u}_k = (q_l\bar{u}_k)_{l\in K} \qquad (*)$$

Let us suppose $s \in S$. Then $s = g(q_ks)_{k\in K}$ according to the axiom (*c*). Thus, $g^{-1}s = (q_ks)_{k\in K}$ and

$$p_kg^{-1}s = q_ks \qquad (**)$$

Therefore, $o_kp_kg^{-1}s = o_kq_ks$. We have $q_ks \in q_kS$; thus, $o_kq_ks = (q_lq_ks)_{l\in K}$ according to (*). It follows $go_kp_kg^{-1}s = g(q_lq_ks)_{l\in K} = q_ks$ according to the axiom (*c*). Thus, the unary operation $go_kp_kg^{-1}$ of $\boldsymbol{A}(S, (q_kS)_{k\in K}, g, n)$ coincides with q_k.

Let us have an arbitrary element $(s_k)_{k\in K} \in \bigtimes_{k\in K} q_kS$. Thus, $q_ks_k = s_k$ for every $k \in K$ according to 1.5 (*i*). It follows $g(s_k)_{k\in K} = g(q_ks_k)_{k\in K} = g(p_kg^{-1}s_k)_{k\in K}$ according to (**). Thus, the partial operation of $\boldsymbol{A}(S,(q_kS)_{k\in K}, g, n)$ coincides with g.

Thus, we have proved that $go_kp_kg^{-1} = q_k$ for every $k \in K$ and that $g(p_kg^{-1}s_k)_{k\in K} = g(s_k)_{k\in K}$ for every $(s_k)_{k\in K} \in \bigtimes_{k\in K} q_kS$. According to 1.2 it means that $\boldsymbol{A}(s, (q_kS)_{k\in K}, g, n) = (S, n, (q_k)_{k\in K}, g)$.

1.10. Theorem. *The operator* $\boldsymbol{A}$ *is a surjection of the class of all admissible quadruples onto the class of all C-algebras.*

2. SUBALGEBRAS OF CARTESIAN ALGEBRAS

2.1. Remark. Let S, K be sets, f a partial operation of type K on the set S. Then f can be considered as a subset of $S^K \times S$. If $T \subseteq S$ then we put $f \mid T = f \cap (T^K \times S)$. Clearly, $f \mid T$ is a mapping defined on a subset of T^K with values in S.

2.2. Theorem. *Every subalgebra of a C-algebra is a C-algebra.*

Proof. Let $(S, n, (q_k)_{k\in K}, g)$ be a C-algebra, $T \subseteq S$ a closed subset.

Then $n \in T$, $q_k T \subseteq T$ for every $k \in K$ and $g(\mathop{\times}_{k\in K} (q_k S \cap T)) \subseteq T$. Let k, $l \in K$. Then

$$(q_k \mid T)(q_l \mid T) = q_k q_l \mid T = \begin{cases} q_k \mid T \text{ for } k = l \\ n \text{ for } k \neq l, \end{cases}$$

which is the axiom (a).

Clearly, $q_k T \subseteq q_k S \cap T$ for every $k \in K$. Let us suppose $k \in K$, $s_k \in q_k S \cap T$. Then $s_k = q_k s_k \in q_k T$ according to 1.5 (i). Thus, $q_k S \cap T = q_k T = (q_k \mid T)\, T$. It follows dom $(g \mid T) = \mathop{\times}_{k\in K} (q_k S \cap T) = \mathop{\times}_{k\in K} (q_k \mid T)\, T$ which is the axiom (b).

For an arbitrary $t \in T$ we have $(g \mid T)\, ((q_k \mid T)\, t)_{k\in K} = g(q_k t)_{k\in K} = t$ which is the axiom (c).

Let us suppose $k \in K$ and $(t_l)_{l\in K} \in \mathop{\times}_{l\in K} (q_l \mid T)\, T = \mathop{\times}_{l\in K} q_l T$. Then $(q_k \mid T)$ $(g \mid T)(t_l)_{l\in K} = q_k g(t_l)_{l\in K} = t_k$ which is the axiom (d).

3. PAIRS OF CARTESIAN ALGEBRAS

3.1. Remark. Two mappings f, f' whose domains are subsets of a set A are considered to be equal iff dom $f =$ dom f' and $fx = f'x$ for every $x \in$ dom f. We write, in this case, $f = f'$.

3.2. Definition. Let us have C-algebras $(S, n, (q_k)_{k\in K}, g)$, $(S, n, (q'_{k'})_{k'\in K'}, g')$. Let $s_{kk'}$ be an element of S for every $(k, k') \in K \times K'$. We put $g' \circ g(s_{kk'})_{(k,k')\in K\times K'} = g'(g(s_{kk'})_{k\in K})_{k'\in K'})$ iff the right side member is defined. Otherwise, $g' \circ g(s_{kk'})_{(k,k')\in K\times K'}$ is considered to be undefined.

3.3. Theorem. *Let $(S, n, (q_k)_{k\in K}, g)$, $(S, n, (q'_{k'})_{k'\in K'}, g')$ be C-algebras. Then the following conditions are equivalent:*

(A) $q_k q'_{k'} = q'_{k'} q_k$ for every $(k, k') \in K \times K'$.

(B) $g' \circ g = g \circ g'$.

Proof. 1. Let (A) hold true. Let $s_{kk'} \in S$ be such elements that $g' \circ g(s_{kk'})_{(k,k')\in K\times K'}$ is defined, i.e. $g'(g(s_{kk'})_{k\in K})_{k'\in K'}$ is defined. It follows $s_{kk'} \in q_k S$ for every $(k, k') \in K \times K'$ and $g(s_{kk'})_{k\in K} \in q'_{k'} S$ for every $k' \in K'$. We have $q_k g(s_{lk'})_{l\in K} = s_{kk'}$ for every $(k, k') \in K \times K'$. It follows $q'_{k'} s_{kk'} = q'_{k'} q_k g(s_{lk'})_{l\in K} = q_k q'_{k'} g(s_{lk'})_{l\in K}$; we have $q'_{k'} g(s_{lk'})_{l\in K} = g(s_{lk'})_{l\in K}$ according to 1.5 (i). It follows $q'_{k'} s_{kk'} = q_k g(s_{lk'})_{l\in K} = s_{kk'}$, i.e. $s_{kk'} \in q'_{k'} S$ and $s_{kk'} \in q_k S \cap q'_{k'} S$. Therefore $g'(s_{kk'})_{k'\in K'}$ is defined. We have $q'_{k'} q_k g'(s_{kl'})_{l'\in K'} = q_k q'_{k'} g'(s_{kl'})_{l'\in K'} = q_k s_{kk'} = s_{kk'}$ according to 1.5 (i). Thus, $g'(s_{kk'})_{k'\in K'} = g'(q'_{k'} q_k g'(s_{kl'})_{l'\in K'})_{k'\in K'} = q_k g'(s_{kl'})_{l'\in K'} \in q_k S$. It follows that $g(g'(s_{kk'})_{k'\in K'})_{k\in K}$ is defined. Thus dom $g' \circ g \subseteq$ dom $g \circ g'$. Similarly, we prove dom $g \circ g' \subseteq$ dom $g' \circ g$. It follows that dom $g' \circ g =$ dom $g \circ g'$.

Let us put $s = g'(g(s_{kk'})_{k\in K})_{k'\in K'}$. We have $q'_{k'}s = g(s_{kk'})_{k\in K}$ for every $k' \in K'$. It follows $q_k q'_{k'} s = q_k g(s_{lk'})_{l\in K} = s_{kk'}$. Thus, $q'_{k'} q_k s = s_{kk'}$ and $g(g'(s_{kk'})_{k'\in K'})_{k\in K} = g(g'(q'_{k'} q_k s)_{k'\in K'})_{k\in K} = g(q_k s)_{k\in K} = s$.

We have proved $g' \circ g = g \circ g'$. Thus, (B) holds true.

2. Let (B) hold true. Let us suppose $s \in S$, $(k, k') \in K \times K'$. We have $g(q_k q'_{k'} s)_{k\in K} = q'_{k'} s$. It follows $g \circ g'(q_k q'_{k'} s)_{(k, k')\in K\times K'} = g' \circ g(q_k q'_{k'} s)_{(k, k')\in K\times K'} = g'(g(q_k q'_{k'} s)_{k\in K})_{k'\in K'} = g'(q'_{k'} s)_{k'\in K'} = s$. It follows $q_k s = q_k g \circ g'(q_l q'_{k'} s)_{(l, k')\in K\times K'} = q_k g(g'(q_l q'_{k'} s)_{k'\in K'})_{l\in K} = g'(q_k q'_{k'} s)_{k'\in K'}$ and, from the last equation, we get $q'_{k'} q_k s = q'_{k'} g'(q_k q'_{l'} s)_{l'\in K'} = q_k q'_{k'} s$. Thus, (A) holds true.

3.4. Lemma. *Let $(S, n, (q_k)_{k\in K}, g)$, $(S, n, (q'_{k'})_{k'\in K'}, g')$ be C-algebras. If the condition (A) of* 3.3 *is fulfilled then the set $q'_{k'}S$ is closed in $(S, n, (q_k)_{k\in K}, g)$ for every $k' \in K'$.*

Proof. According to 1.5 (ii) we have $n = q'_{k'} n \in q'_{k'} S$ and $q_k q'_{k'} S = q'_{k'} q_k S \subseteq q'_{k'} S$; thus $q'_{k'} S$ is closed with respect to the nullary and unary operations.

Let us have $s_k \in q_k S \cap q'_{k'} S$ for every $k \in K$. We put $s = g(s_k)_{k\in K}$. Then we have $q_k q'_{k'} s = q'_{k'} q_k s = q'_{k'} q_k g(s_l)_{l\in K} = q'_{k'} s_k = s_k$ according to 1.5 (i). Thus, $s = g(s_k)_{k\in K} = g(q_k q'_{k'} s)_{k\in K} = q'_{k'} s$ and $s \in q'_{k'} S$. We have proved that $g \mid q'_{k'} S$ is a partial operation with values in $q'_{k'} S$.

3.5. Definition. Let K, K', S be non-empty sets, let $U_{kk'}$ be a set for every $(k, k') \in K \times K'$, U_k a set for every $k \in K$. Let $f\colon \bigtimes_{k\in K} U_k \to S$ be a mapping, $f'_k\colon \bigtimes_{k'\in K'} U_{kk'} \to U_k$ a mapping for every $k \in K$. Then $f(f'_k)_{k\in K}$ is the mapping defined on $\bigtimes_{(k, k')\in K\times K'} U_{kk'}$ in the following way: for every $(u_{kk'})_{(k, k')\in K\times K'} \in \bigtimes_{(k, k')\in K\times K'} U_{kk'}$ we put $f(f'_k)_{k\in K}(u_{lk'})_{(l, k')\in K\times K'} = f(f'_k(u_{kk'})_{k'\in K'})_{k\in K}$.

3.6. Lemma. *Let $(S, n, (q_k)_{k\in K}, g)$, $(S, n, (q'_{k'})_{k'\in K'}, g')$ be C-algebras. If the condition (A) of* 3.3 *is fulfilled then the following condition is fulfilled, too:*

(C) For every $k \in K$ there exists a C-algebra $(q_k S, n, (q'_{kk'})_{k'\in K'}, g'_k)$ and for every $k' \in K'$ there exists a C-algebra $(q'_{k'} S, n, (q_{kk'})_{k\in K}, g_{k'})$ such that $g(g'_k)_{k\in K} = g'(g_{k'})_{k'\in K'}$.

Proof. The set $q_k S$ is closed in $(S, n, (q_{k'})_{k'\in K'}, g')$ for every $k \in K$ according to 3.4. If we put $q'_{kk'} = q'_{k'} \mid q_k S$ for every $k' \in K'$, $g'_k = g' \mid q_k S$ then $(q_k S, n, (q'_{kk'})_{k'\in K'}, g'_k)$ is a subalgebra of $(S, n, (q'_{k'})_{k'\in K'}, g')$ which is a C-algebra according to 2.2. In a similar way we define $(q'_{k'} S, n, (q_{kk'})_{k\in K}, g_{k'})$ for every $k' \in K'$.

Now, $g(g'_k)_{k\in K}(s_{lk'})_{(l, k')\in K\times K'} = g(g'_k(s_{kk'})_{k'\in K'})_{k\in K}$ is defined iff

$s_{kk'} \in q'_{kk'}q_kS = (q'_{k'} \mid q_kS)\, q_kS = q'_{k'}q_kS = q_kq'_{k'}S = (q_k \mid q'_{k'}S)\, q'_{k'}S = q_{kk'}q'_{k'}S$ for every $(k, k') \in K \times K'$ which is a necessary and sufficient condition for the existence of $g'(g_{k'}(s_{kk'})_{k \in K})_{k' \in K'}$. We have $g(g'_k(s_{kk'})_{k' \in K'})_{k \in K} = g(g'(s_{kk'})_{k' \in K'})_{k \in K} = g'(g(s_{kk'})_{k \in K})_{k' \in K'} = g'(g_{k'}(s_{kk'})_{k \in K})_{k' \in K'}$ according to **3.3** which means $g(g'_k)_{k \in K} = g'(g_{k'})_{k' \in K'}$.
Thus, (C) holds true.

3.7. Lemma. *Let $(S, n, (q_k)_{k \in K}, g)$, $(S, n, (q'_{k'})_{k' \in K'}, g')$ be C-algebras for which the condition (C) of* **3.6** *ist fulfilled. Then the condition (A) of* **3.3** *ist fulfilled for them, too.*

Proof. 1. Let $s \in S$ be an arbitrary element. We have $s = g(q_ks)_{k \in K}$, $q_ks = g'_k(q'_{kk'}q_ks)_{k' \in K'}$ for every $k \in K$. If we put $s_{kk'} = q'_{kk'}q_ks$ for every $(k, k') \in K \times K'$ then $s = g(g'_k(s_{kk'})_{k' \in K'})_{k \in K}$. Thus, we have proved, for every $s \in S$, the existence of a system $(s_{kk'})_{(k, k') \in K \times K'}$ with the property $s = g(g'_k(s_{kk'})_{k' \in K'})_{k \in K}$.

2. For an arbitrary $s \in S$ we define the elements $s_{kk'}$ according to the first part of the proof. Then we put $s_{k'} = q'_{k'}s = g_{k'}(s_{kk'})_{k \in K}$. According to 1 there exists a system $(t_{kk'})_{(k, k') \in K \times K'}$ with the property $s_{k'} = g(g'_k(t_{kl'})_{l' \in K'})_{k \in K} = g'(g_{l'}(t_{kl'})_{k \in K})_{l' \in K'}$. It follows $q'_{l'}s_{k'} = g_{l'}(t_{kl'})_{k \in K}$. From $s_{k'} \in q'_{k'}S$ it follows according to **1.5** (i)

$$g_{l'}(t_{kl'})_{k \in K} = q'_{l'}s_{k'} = \begin{cases} n \text{ for } l' \in K',\ l' \neq k' \\ s_{k'} \text{ for } l' \in K',\ l' = k'. \end{cases}$$

For $l' \neq k'$ we have, according to **1,5** (ii), $n = q_{kl'}n = q_{kl'}g_{l'}(t_{ll'})_{l \in K} = t_{kl'}$. According to **1.5** (iii) we have $g'_k(t_{kl'})_{l' \in K'} = t_{kk'}$ for every $k \in K$. It follows $s_{k'} = g(t_{kk'})_{k \in K}$. From the equation $g_{k'}(s_{kk'})_{k \in K} = s_{k'} = g_{k'}(t_{kk'})_{k \in K}$ we get $s_{kk'} = q_{kk'}s_{k'} = t_{kk'}$ for every $k \in K$. Thus, $s_{k'} = g(s_{kk'})_{k \in K}$ and $q_kq'_{k'}s = q_ks_{k'} = s_{kk'}$.

3. In the same way we prove $q'_{k'}q_ks = s_{kk'}$. Thus, we have $q_kq'_{k'}s = q_{k'}q_ks$ for every $s \in S$ and every $(k, k') \in K \times K'$. We have proved that (A) is fulfilled.

3.8. Lemma. *Let S, K, K' be non-empty sets, let U_k be a set for every $k \in K$ and $U'_{k'}$ be a set for every $k' \in K'$. Let $f\colon \bigtimes_{k \in K} U_k \to S$, $f'\colon \bigtimes_{k' \in K'} U'_{k'} \to S$ be bijections, $n \in S$ an arbitrary element. Then the following conditions are equivalent:*

(δ) For every $(k, k') \in K \times K'$ there exists a set $U_{kk'}$, for every $k \in K$ there exists a bijection $f'_k\colon \bigtimes_{k' \in K'} U_{kk'} \to U_k$ and for every $k' \in K'$ there exists a bijection $f_{k'}\colon \bigtimes_{k \in K} U_{kk'} \to U'_{k'}$ such that $f(f'_k)_{k \in K} = f'(f_{k'})_{k' \in K'}$.

(γ) For the C-algebras $\boldsymbol{A}(S, (U_k)_{k \in K}, f, n)$, $\boldsymbol{A}(S, (U'_{k'})_{k' \in K'}, f', n)$ the condition (C) of **3.6** *is fulfilled.*

Proof. I. 1. Let (δ) be fulfilled. According to 1.2 we put $(n_{kk'})_{k'\in K'} = f_k'^{-1} n_k$ for every $k \in K$; if $(u_{kk'})_{k'\in K'} \in \bigtimes_{k'\in K'} U_{kk'}$ we put $p_{kk_0'}(u_{kk'})_{k'\in K'} = u_{kk_0'}$ for every $k_0' \in K'$. For arbitrary $(k, k_0') \in K \times K'$, $\bar{u}_{kk_0'} \in U_{kk_0'}$ we put $o_{kk_0'} u_{kk_0'} = (u_{kk'})_{k'\in K'}$ where $u_{kk'} = n_{kk'}$ for every $k' \in K'$, $k' \neq k_0'$ and $u_{kk_0'} = \bar{u}_{kk_0'}$. The symbols $n_{kk'}'$, $p_{kk'}'$, $o_{kk'}'$ have a similar meaning (instead of f_k' we take $f_{k'}$, instead of n_k the element $n_{k'}'$). We have $f(f_k'(n_{kk'})_{k'\in K'})_{k\in K} = f(n_k)_{k\in K} = n = f'(n_{k'}')_{k'\in K'} = f'(f_{k'}(n_{kk'}')_{k\in K})_{k'\in K'} = f(f_k'(n_{kk'}')_{k'\in K'})_{k\in K}$. As the mappings f, f_k' are injections it follows $f_k'(n_{kk'})_{k'\in K'} = f_k'(n_{kk'}')_{k'\in K'}$ for every $k \in K$, thus $n_{kk'} = n_{kk'}'$ for every $(k, k') \in K \times K'$.

2. The mapping $fo_k f_k'$ is an injection of $\bigtimes_{k'\in K'} U_{kk'}$ onto $fo_k U_k = q_k S$ for every $k \in K$ according to 1.8. Thus, $(q_k S, (U_{k'k})_{k'\in K'}, fo_k f'_k, n)$ is an admissible quadruple and $(q_k S, n, (q'_{kk'})_{k'\in K'}, q'_k) = A((q_k S, (U_{kk'})_{k'\in K'}, fo_k f_k', n)$ is a C-algebra. Similarly, we define $(q_{k'}' S, n, (q_{kk'})_{k\in K}, g_{k'})$ for every $k' \in K'$. According to 1.2 we have $q_{kk'}' = fo_k f_k' o_{kk'} p_{kk'} (fo_k f_k')^{-1} = fo_k f_k' o_{kk'} p_{kk'} f_k'^{-1} p_k f^{-1}$ and similarly $q_{kk'} = f' o_{k'}' f_{k'} o_{kk'}' p'_{kk'} f_{k'}^{-1} p_{k'}' f'^{-1}$.

3. Let us have arbitrary elements $(k_0, k_0') \in K \times K'$, $\bar{u}_{k_0 k_0'} \in U_{k_0 k_0'}$. Let us put $u_{kk'} = n_{kk'}$ for every $(k, k') \in K \times K'$, $(k, k') \neq (k_0, k_0')$ and $u_{k_0 k_0'} = \bar{u}_{k_0 k_0'}$. Then $fo_{k_0} f_{k_0}' o_{kk_0'} \bar{u}_{k_0 k_0'} = fo_{k_0} f_{k_0}' (u_{k_0 k'})_{k'\in K'} = f(f_k'(u_{kk'})_{k'\in K'})_{k\in K} = f'(f_{k'}(u_{kk'})_{k\in K})_{k'\in K'} = f' o_{k_0'}' f_{k_0'} (u_{kk_0'})_{k\in K} = f' o'_{k_0'} f_{k_0'} o_{k_0 k_0'}' \bar{u}_{k_0 k_0'}$ according to 1. Thus, $fo_k f_k' o_{kk'} = f' o_{k'}' f_{k'} o_{kk'}'$ for every $(k, k') \in K \times K'$. It follows $q_{kk'}' q_k S = fo_k f_k' o_{kk'} p_{kk'} f_k'^{-1} p_k f^{-1} fo_k p_k f^{-1} S = fo_k f_k' o_{kk'} p_{kk'} f_k'^{-1} p_k f^{-1} S = fo_k f_k' o_{kk'} U_{kk'}$ according to 2. Similarly we prove that $q_{kk'} q_{k'}' S = f' o_{k'}' f_{k'} o_{kk'}' U_{kk'}$. Thus, $q_{kk'}' q_k S = fo_k f_k' o_{kk'} U_{kk'} = f' o_{k'}' f_{k'} o_{kk'} U_{kk'} = q_{kk'} q_{k'}' S$ for every $(k, k') \in K \times K'$.

4. Let us suppose $s_{kk'} \in S$ for every $(k, k') \in K \times K'$. Then $g(g_k'(s_{kk'})_{k'\in K'})_{k\in K}$ is defined iff $s_{kk'} \in q_{kk'}' q_k S = q_{kk'} q_{k'}' S$ (cf. 3) which is a necessary and sufficient condition for the existence of $g'(g_{k'}(s_{kk'})_{k\in K})_{k'\in K'}$. In this case $s_{kk'} \in fo_k f_k' o_{kk'} U_{kk'}$ which means the existence of an element $u_{kk'} \in U_{kk'}$ with the property $s_{kk'} = fo_k f_k' o_{kk'} u_{kk'} = f' o_{k'}' f_{k'} o_{kk'}' u_{kk'}$. According to 1.2 we have $g_k'(s_{kk'})_{k'\in K'} = fo_k f_k' (p_{kk'} f_k'^{-1} p_k f^{-1} s_{kk'})_{k'\in K'} = fo_k f_k' (p_{kk'} f_k'^{-1} p_k f^{-1} fo_k f_k' o_{kk'} u_{kk'})_{k'\in K'} = fo_k f_k' (u_{kk'})_{k'\in K'}$. Thus, $g(g_k'(s_{kk'})_{k'\in K'})_{k\in K} = f(p_k f^{-1} fo_k f_k' (u_{kk'})_{k'\in K'})_{k\in K} = f(f_k'(u_{kk'})_{k'\in K'})_{k\in K}$ according to 1.3 (i). Similarly, we prove $g'(g_{k'}(s_{kk'})_{k\in K})_{k'\in K'} = f'(f_{k'}(u_{kk'})_{k\in K})_{k'\in K'}$. Thus, we have $g(g_k'(s_{kk'})_{k'\in K'})_{k\in K} = f(f_k'(u_{kk'})_{k'\in K'})_{k\in K} = f'(f_{k'}(u_{kk'})_{k\in K})_{k'\in K'} = g'(g_{k'}(s_{kk'})_{k\in K})_{k'\in K'}$ and the condition (C) of 3.6 is fulfilled for the C-algebras $A(S, (U_k)_{k\in K}, f, n)$, $A(S, (U_{k'}')_{k'\in K'}, f', n)$. Thus (γ) holds true.

II. Let (γ) be fulfilled. We put $A(S, (U_k)_{k\in K}, f, n) = (S, n, (q_k)_{k\in K}, g)$, $A(S, (U_{k'}')_{k'\in K'}, f', n) = (S, n, (q_{k'}')_{k'\in K'}, g')$. There exist C-algebras $(q_k S, n, (q_{kk'}')_{k'\in K'}, g'_k)$ for every $k \in K$ and $(q_{k'}' S, n, (q_{kk'})_{k\in K}, g_{k'})$ for every $k' \in K'$ such that $g(g'_k)_{k\in K} = g'(g_{k'})_{k'\in K'}$. It follows from the last equation that $q'_{kk'}q_kS = q_{kk'}q'_{k'}S$ for every $(k, k') \in K \times K'$. We put $U_{kk'} = q'_{kk'}q_kS = q_{kk'}q'_{k'}S$ for every $(k, k') \in K \times K'$, $f'_k = p_kf^{-1}g'_k$ for every $k \in K$, $f_{k'} = p'_{k'}f'^{-1}g_{k'}$ for every $k' \in K'$. According to 1.8 and 1.9 g'_k is a bijection of $\bigtimes_{k'\in K'} U_{kk'}$ onto $q_kS = fo_kU_k$ for every $k \in K$. Thus, p_kf^{-1} is a bijection of q_kS onto U_k for every $k \in K$. It follows that f'_k is a bijection of $\bigtimes_{k'\in K'} U_{kk'}$ onto U_k for every $k \in K$.

Let us suppose $u_{kk'} \in U_{kk'}$ for every $(k, k') \in K \times K'$. Then $f(f'_k(u_{kk'})_{k'\in K'})_{k\in K} = f(p_kf^{-1}g'_k(u_{kk'})_{k'\in K'})_{k\in K} = g(g'_k(u_{kk'})_{k'\in K'})_{k\in K}$ according to 1.2. Similarly, we have $f'(f_{k'}(u_{kk'})_{k\in K})_{k'\in K'} = g'(g_{k'}(u_{kk'})_{k\in K})_{k'\in K'}$. It follows from our suppositions that $f(f'_k)_{k\in K} = g(g'_k)_{k\in K} = g'(g_{k'})_{k'\in K'} = f'(f_{k'})_{k'\in K'}$. Thus, the condition (δ) holds true.

4. MAIN THEOREM

From the results of the paragraph **3** we get the following theorem:

4.1. Theorem. *Let S, K, K' be non-empty sets, let U_k be a set for every $k \in K$ and $U'_{k'}$ a set for every $k' \in K'$. Let $f\colon \bigtimes_{k\in K} U_k \to S$, $f'\colon \bigtimes_{k'\in K'} U'_{k'} \to S$ be bijections, $n \in S$ an arbitrary element. Then the following assertions are equivalent:*

(δ) *For every $(k, k') \in K \times K'$ there exists a set $U_{kk'}$, for every $k \in K$ there exists a bijection $f'_k\colon \bigtimes_{k'\in K'} U_{kk'} \to U_k$ and for every $k' \in K'$ there exists a bijection $f_{k'}\colon \bigtimes_{k\in K} U_{kk'} \to U'_{k'}$ such that $f(f'_k)_{k\in K} = f'(f_{k'})_{k'\in K'}$.*

(α) *For the C-algebras $\boldsymbol{A}(S, (U_k)_{k\in K}, f, n) = (S, n, (q_k)_{k\in K}, g)$, $\boldsymbol{A}(S, (U'_{k'})_{k'\in K'}, f', n) = (S, n, (q'_{k'})_{k'\in K'}, g')$ we have $q_kq'_{k'} = q'_{k'}q_k$ for every $(k, k') \in K \times K'$.*

(β) *For the C-algebras $\boldsymbol{A}(S, (U_k)_{k\in K}, f, n) = (S, n, (q_k)_{k\in K}, g)$, $\boldsymbol{A}(S, (U'_{k'})_{k'\in K'}, f', n) = (S, n, (q'_{k'})_{k'\in K'}, g')$ we have $g' \circ g = g \circ g'$.*

(γ) *For the C-algebras $\boldsymbol{A}(S, (U_k)_{k\in K}, f, n) = (S, n, (q_k)_{k\in K}, g)$, $\boldsymbol{A}(S, (U'_{k'})_{k'\in K'}, f', n) = (S, n, (q'_{k'})_{k'\in K'}, g')$ the following condition is fulfilled: For every $k \in K$ there exists a C-algebra $(q_kS, n, (q'_{kk'})_{k'\in K'}, g'_k)$ and for every $k' \in K'$ there exists a C-algebra $(q'_{k'}S, n, (q_{kk'})_{k\in K}, g_{k'})$ such that $g(g'_k)_{k\in K} = g'(g_{k'})_{k'\in K'}$.*

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