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ON ONE ALGORITHM FINDING ALL BIMATRIX GAME EQUILIBRIA

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To Professor O. Borůvka, on his 70th birthday

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Consider a two-person noncooperative game $\Gamma = \{\{1, 2\}, (A_1, A_2), (f_1, f_2)\}$ with real payoffs[1]) (denote by A_i the finite set of all i's pure strategies, by S_i the A_i's probability simplex, construct cartesian products $A =: A_1 \otimes A_2$ and $S =: S_1 \otimes S_2$, prolong the function f_i from vert S on the whole S (in a natural way) and denote by $E \subset S$ the set of all Nash Γ's equilibria ($(\bar{x}_1, \bar{x}_2) \in E$ iff for all $x_1 \in S_1$, $x_2 \in S_2$ it holds $f_1(x_1, \bar{x}_2) \leqq f_1(\bar{x}_1, \bar{x}_2), f_2(\bar{x}_1, x_2) \leqq f_2(\bar{x}_1, \bar{x}_2)$ — see [4])). We say Φ_i is σ_i-inclusive (see [6] or [5]) if Φ_i maps S_i into $\mathscr{F}(S_{i+1})$ (i mod 2) in such a way that it holds $[x_i, y_i \in \operatorname{relint} T, T \in \mathscr{F}(\sigma_i) \Rightarrow \Phi_i(x_i) = \Phi_i(y_i)]$ and $[x_i \in \operatorname{relint} L, y_i \in \operatorname{relint} T, L, T \in F(\sigma_i), L \subset T \Rightarrow$ either $\Phi_i(x_i) \subset \subset \Phi_i(y_i)$ or $\Phi_i(x_i) \supset \Phi_i(y_i)]$, where σ_i is a polyhedral partition of S_i. Define Φ_i on S_i by $\Phi_i(x_i) = \{x_{i+1} \in S_{i+1} \mid f_{i+1}(x_i, x_{i+1}) \geqq f_{i+1}(x_i, v^j)$ for all $v^j \in \operatorname{vert} S_{i+1}\}$ (i mod 2), construct $R_j = \{(x_i, z) \in (\operatorname{aff} S_i) \otimes \mathbf{E}^1 \mid x_i \in \in \operatorname{aff} S_i, z \in \mathbf{E}^1, z \geqq f_{i+1}(x_i, v^j)\}$ for each $v^j \in \operatorname{vert} S_{i+1}$ and put $R = = : \bigcap_j R_j$. Evidently R_j's are halfspaces in $\mathbf{E}^{\operatorname{card} A_i}$, R a polyhedral set, the orthogonal projection of R's boundary (into aff S_i) is a polyhedral partition of aff S_i (its intersection with S_i denote by σ_i) and Φ_i is σ_i-inclusive. Evidently $(\bar{x}_1, \bar{x}_2) \in E$ iff Φ_1, Φ_2 have in $(\bar{x}_1, \bar{x}_2)$ a coincidence (i.e. $\bar{x}_1 \in \Phi_2(\bar{x}_2)$, $\bar{x}_2 \in \Phi_1(x_1)$). Choose for each $T \in \mathscr{F}(\sigma_i)$ one point $\tau_i(T) \in \in \operatorname{relint} T$ and the set of all $\tau_i(T)$'s denote by X_i (i.e. $\tau_i(T)$ one-one maps $\mathscr{F}(\sigma_i)$ onto X_i). Because of $[x_1, \bar{x}_1 \in \operatorname{relint} T, x_2, \bar{x}_2 \in \operatorname{relint} L, T \in \mathscr{F}(\sigma_1), L \in \mathscr{F}(\sigma_2), (\bar{x}_1, \bar{x}_2) \in E \Rightarrow (x_1, x_2) \in E]$ (because Φ_i is σ_i-inclusive and it holds $[(\operatorname{relint} U) \cap \Phi_i(x_i) \neq \emptyset, U \in \mathscr{F}(S_{i+1}) \Rightarrow U \in \mathscr{F}(\Phi_i(x_i))]$) and $E \neq \emptyset$ (see [4]), E being closed, we have proved the following statement

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1) A Euclidean n-dimensional space denote by $\mathbf{E}^n$ and the smallest space containing $X \subset \mathbf{E}^n$ by aff X. A nonvoid intersection of a finite number of halfspaces is called a polyhedral set (say P), vert P is the set of all its vertices, $\mathscr{F}(P)$ the set of all its nonvoid faces (of all dimensions k, $0 \leqq k \leqq \dim P$), relint P the set of all its inner (in the space aff P) points (for $x \in \mathbf{E}^n$ it is relint $\{x\} = \{x\}$). For $X \subset \mathbf{E}^n$, $\dim X = n$, a polyhedral partition σ of X is a finite set of n-dimensional polyhedral sets P_i's such that $\bigcup P_i = X$ and $[P_i \cap P_j \neq \emptyset, P_i, P_j \in \sigma \Rightarrow P_i \cap P_j \in \mathscr{F}(P_i) \cap \cap \mathscr{F}(P_j)]$. The set of all P_i's faces (for all dimensions k, $0 \leqq k \leqq n$, and all $P_i \in \sigma$) is denoted by $\mathscr{F}(\sigma)$.

(the constructions of $\mathscr{F}(\sigma_1)$, $\mathscr{F}(\sigma_2)$, X_1, X_2, $(X_1 \otimes X_2) \cap E$ and B are simple linear programming tasks):

Theorem:²) $E = \bigcup [\tau_1^{-1}(x_1) \otimes \tau_2^{-1}(x_2)]$ *where the sum operates on the set* B *of all* $(x_1, x_2) \in (X_1 \otimes X_2) \cap E$ *with maximal (in* $\subset$ *)* $\tau_1^{-1}(x_1) \otimes \tau_2^{-1}(x_2)$.

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²) Several constructions of E are known (see [7], [3], [1], [2]). This paper presents (using ideas of [5] and [6]) the "inclusive" approach to the results of [7].