Archivum Mathematicum

František Neuman

Note on Kummer's transformation

Archivum Mathematicum, Vol. 6 (1970), No. 3, 185--188

Persistent URL: http://dml.cz/dmlcz/104721

NOTE ON KUMMER'S TRANSFORMATION

F. NEUMAN, BRNO

(Received August 27, 1969)

I. The transformation

$$K : Y(T) \mapsto y(t) = w(t) \,.\, Y(X(t))$$

which maps any solution $Y(T)$ of a differential equation

$$\text{(Q)} \qquad \left(\frac{d^2Y}{dT^2} =\right) \qquad \ddot{Y} = Q(T)\, Y \text{ on } (A,B)$$

in a solution $y(t)$ of a differential equation

$$\text{(q)} \qquad \left(\frac{d^2y}{dt^2} =\right) \qquad y'' = q(t)\, y \text{ on } (a, b)$$

was first considered by E. E. Kummer [1] in 1834. He discovered that then $w(t) = C \,.\, | X'(t) |^{-1/2}$, C — a constant, and that there holds

$$\text{(Qq)} \qquad -\{X, t\} + Q(X)\, X'^2(t) = q(t),$$

where

$$\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\left(\frac{X''(t)}{X'(t)}\right)^2$$

is Schwarz's derivative of $X(t)$.

However, the deep study of this (Kummer's) transformation, the determination of the interval of the definition of such transformation, the necessary and sufficient condition for the existence of a solution $X(t)$ of (Kummer's) equation (Qq) was given only recently by Prof. O. Borůvka in his articles and mainly in his book on differential transformations [2].

In all these investigations the following conditions were supposed: $w(t) \neq 0$, $X'(t) \neq 0$, $X(t) \in C^3$, for all t for which Kummer's transformation is defined.

In this note the consequences of some weaker suppositions will be considered.

Further everywhere, if Kummer's transformation is considered, then only the existence of functions $w(t)$ and $X(t)$ on (a, b), and $(A, B) \supset \supset \{X(t),\, t \in (a, b)\}$ are supposed.

II. It holds:

Theorem 1. *If Kummer's transformation*

$$K : Y(T) \mapsto y(t) = w(t) \,.\, Y(X(t))$$

maps all solutions $Y(T)$ of (Q) on (A, B) onto all solutions $y(t)$ of (q) on (a, b), or equivalently,

if Kummer's transformation $\boldsymbol{K}$ *transforms some two linearly independent solutions Y_1 and Y_2 of (Q) on (A, B) into again linearly independent solutions $y_1 = \boldsymbol{K}(Y_1)$ and $y_2 = \boldsymbol{K}(Y_2)$ of (q) on (a, b),*

then $w(t) \neq 0$, $X'(t) \neq 0$, $X(t) \in C^3$ everywhere on (a, b).

Proof. First we show the equivalence of those two assumptions.

a) If any pair of linearly independent solutions Y_1, Y_2 is transformed into linearly dependent solutions, then Kummer's transformations of all solutions of (Q) are linearly dependent, and hence we cannot obtain all solutions of (q).

b) If, for linearly independent Y_1 and Y_2, $\boldsymbol{K}(Y_1) = y_1$ and $\boldsymbol{K}(Y_2) = y_2$ are again linearly independent, then to every solution y of (q) ($y = c_1 y_1 + c_2 y_2$) there exists a solution Y of (Q) ($Y = c_1 Y_1 + c_2 Y_2$) for which $\boldsymbol{K}(Y) = y$.

Now, let Y_1 and Y_2 be linearly independent solutions of (Q) on (A, B), and the same hold for $\boldsymbol{K}(Y_1) = y_1$ and $\boldsymbol{K}(Y_2) = y_2$ of (q) on (a, b).

Then $w(t) \neq 0$ on (a, b). This being not the case and $w(t_0) = 0$ for $t_0 \in (a, b)$, then $y_1(t_0) = w(t_0)\, Y_1(X(t_0)) = 0$ and $y_2(t_0) = w(t_0)\, Y_2(X(t_0)) = 0$. Thus $y_1(t)$ and $y_2(t)$ are linearly dependent, which is a contradiction.

Let $t_0 \in (a, b)$. Without lost of generality, let $y_2(t_0) \neq 0$. Then

$$\frac{y_1(t)}{y_2(t)} - \frac{Y_1(X(t))}{Y_2(X(t))} = 0 \tag{1}$$

holds in an neighbourhood V of t_0, and $Y_2(X_0) \neq 0$ for $X_0 = X(t_0)$.

Put

$F(t, X) = y_1(t)/y_2(t) - Y_1(X)/Y_2(X)$ for $(t, X) \in V \times (X_0 - \delta, X_0 + \delta)$, δ — a suitable positive number such that $Y(X) \neq 0$ for $X \in (X_0 - \delta, X_0 + \delta)$. Then $F(t_0, X_0) = 0$,

$$\frac{\partial F(t, X)}{\partial X} = \frac{Y_1(X)\dot{Y}_2(X) - \dot{Y}_1(X)\, Y_2(X)}{Y_2^2(X)} = \frac{W}{Y_2^2(X)} \neq 0$$

and continuous on $V \times (X_0 - \delta, X_0 + \delta)$, and

$$\frac{\partial F(t, X)}{\partial t} = \frac{-W}{y_2^2(t)}$$

is continuous on $V \times (X_0 - \delta, X_0 + \delta)$, where W, w are constants.

Hence $X'(t)$ exists and is continuous on V. According to (1), one has

$$X'(t) = \frac{w}{W} \cdot \frac{Y_2^2(X(t))}{y_2^2(t)} \neq 0 \text{ for } t \in V. \tag{2}$$

As $y_2(t) \in C^2$ and $Y_2(T) \in C^2$, the relation (2) gives $X(t) \in C^3$ in V.

Summarizing, we have got $w(t) \neq 0$, $X'(t) \neq 0$ and continuous $X'''(t)$ for all $t \in (a, b)$, Q.E.D.

III. Let Kummer's transformation of a pair (then any pair) of linearly independent solutions Y_1, Y_2 of (Q) be linearly dependent.

Hence

$$w(t) \,.\, Y_1(X(t)) = c_1 y(t),$$

and

$$w(t) \,.\, Y_2(X(t)) = c_2 y(t),$$

or

$$w(t)\,[c_2 Y_1(X(t)) - c_1 Y_2(X(t))] \equiv 0 \text{ on } (a, b). \tag{3}$$

As $Y_3(X) = c_2 Y_1(X) - c_1 Y_2(X)$ is a solution of (Q) for $X \in (A, B)$, then $c_2 Y_1(X(t)) - c_1 Y_2(X(t)) = 0$ only for such $t \in (a, b)$ for which $X(t)$ is a zero of Y_3.

Let $N \equiv \{t;\ t \in (a, b) \text{ and } X(t) \text{ is a zero of } Y_3\}$. Then, with respect to (3), $w(t) \equiv 0$ on $(a, b) - N \equiv M$. If M is not a subset of the set of zeros of a solution $(y(t))$ of (q), then $c_1 = c_2 = 0$ and Kummer's transformation of any solution of (Q) is the trivial solution (of (q)). Then $w(t) \equiv 0$ on (a, b). Because if this is not the case, and $w(t_0) \neq 0$ for $t_0 \in (a, b)$, one can consider a solution $\bar{Y}$ of (Q) for which $\bar{Y}(X(t_0)) \neq 0$. Then $\bar{Y}$ can be expressed as $\bar{Y} = d_1 Y_1 + d_2 Y_2$ and we get the following contradiction:

$0 \neq w(t_0)\ \bar{Y}(X(t_0)) = w(t_0)\,(d_1 Y_1 + d_2 Y_2) = d_1 c_1 y(t) + d_2 c_2 y(t) = 0$, as $c_1 = c_2 = 0$.

Conversely, let $N \subset (a, b)$ and $M \subset (a, b)$ be given, such that $N \cap M = \varnothing$, $N \cup M = (a, b)$. Further, let $\{X(t);\ t \in N\}$ be a subset of a set N^* of conjugate points of (Q). A set $S \in (A, B)$ is called a set of conjugate points of a differential equation (Q) on (A, B) iff there exists a non-trivial solution of (Q) which vanishes at all points of S and only at them (see [2, p. 15]). Moreover, let M be a subset of a set M^* of conjugate points of (q). Denote by Y_3 the solution of (Q), for which N^* is the set of its zeros. Let $y(t)$ be the solution of (q) having M^* as the set of its zeros. If Y_1 and Y_2 are linearly independent solutions of (Q), let, without loss of generality, Y_1 and Y_3 be linearly independent. Then $Y_1(X(t)) \neq 0$ for $t \in N$ and we can define

$$w(t) = \begin{cases} y(t)/Y_1(X(t)), & t \in N \\ 0\qquad\quad\ , & t \in M. \end{cases}$$

Hence $w(t) \,.\, Y_1(X(t)) = y(t)$ for $t \in (a, b)$. Let $Y_3 = k_1 Y_1 + k_2 Y_2$. For $t \in N$, we have $k_1 Y_1(X(t)) + k_2 Y_2(X(t)) = 0$, or

$$Y_2(X(t)) = \frac{k_1}{k_2}\, Y_1(X(t));$$

(the linear independence of Y_1 and Y_3 gives $k_2 \neq 0$). Thus

$$w(t)\,Y_2(X(t)) = -\frac{k_1}{k_2}w(t)\,Y_1(X(t)) = -\frac{k_1}{k_2}y(t) \quad \text{for} \quad t \in N,$$

and

$$w(t)\,Y_2(X(t)) = 0 = -\frac{k_1}{k_2}\,y(t) \qquad \text{for } t \in M.$$

Hence, also

$$w(t)\,Y_2(X(t)) = cy(t) \quad \text{for} \quad t \in (a, b),\ c = -k_1/k_2.$$

Now, we can summarize our considerations in

Theorem 2. *Let Kummer's transformation* **K** *transform solutions of* (*Q*) *on* (A, B) *into solutions of* (q) *on* (a, b). *Then only the following three cases are possible*:

1. *the assumptions of Theorem 1 are satisfied, and then* $w(t) \neq 0$, $X'(t) \neq 0$, $X(t) \in C^3$ *for all* $t \in (a, b)$ [*and then necessarily* $w(t) = C \cdot |X'(t)|^{-1/2}$, $X(t)$ *satisfying Kummer's equation* (Qq) *on* (a, b)].

2. *the image of this transformation is just one-parametric set of dependent solutions* ($cy(t)$, $y(t) \neq 0$) *of* (q). *This can occur if and only if there exists a set* N^* *of conjugate points of* (Q) *and a subset* M *of* (a, b) *such that* $N^* \supset \{X(t);\ t \in M\}$ *and the set* $(a, b) - M$ *is a subset of a set of conjugate points of* (q).

3. *the image of this transformation* **K** *is just the trivial solution of* (q). *This can occur if and only if* $w(t) \equiv 0$ *on* (a, b).

Corollary. Let **K** be Kummer's transformation of solutions of (Q) on (A, B) into solutions of (q) on (a, b). Let, moreover, $w(t) \neq 0$ on (a, b, and the range of $X(t)$, $t \in (a, b)$, be an infinitive, not enumerable set (e.g. let it contain an open interval $(A_1, B_1) \subset (A, B)$). Then $w(t) \neq$ o, $X'(t) \neq 0$, $X(t) \in C^3$ for all $t \in (a, b)$.

Proof. Since $w(t) \neq 0$, case 3 (in Theorem 2) cannot occur. Now, any set M of conjugate points of (q) is at most enumerable. If case 2 (in Theorem 2) takes place, then the values of $X(t)$ form again at most enumerable set for $t \in (a, b) - M$, $M \subset M^*$ (for notation, see Theorem 2), Hence, the set of values of $X(t)$, for $t \in (a, b)$, is at most enumerable. which is a contradiction. Thus case 1 holds, which leads, according to Theorem 1, to the assertion of the corollary.

REFERENCES

[1] Kummer, E. E.: *De generali quadam aequatione differentiali tertii ordinis*, in „Programm des evangelischen Königl. und Stadtgymnasiums in Liegnitz, 1837, or reprinted in J. reine angew. Math. (Vol. 100), 1887.

[2] Borůvka, O.: *Lineare Differentialtransformationen 2. Ordnung*, VEB, Berlin 1967.

Department of Mathematics
J. E. Purkyně University, Brno
Czechoslovakia