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Milan Kolibiar

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ON A CONSTRUCTION OF SEMIGROUPS

M. KOLIBIAR, Bratislava

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In [2] a construction of certain semigroups is given. This construction starts from semilattices and gives the class of "idempotent right quasi-Abelian semigroups". In the present paper we use an analogous construction which can be started from an arbitrary semigroup. The main result is that any semigroup whose congruence relations satisfy the ascending chain condition (ACC) can be derived from a left reductive semigroup by a finite number of the said constructions. Moreover the least congruence relation on a semigroup is given such that the corresponding quotient semigroup is left reductive.

1. BASIC DEFINITIONS AND LEMMAS

A semigroup S is called left reductive [1] if whenever $xa = xb$ for each $x \in S$ then $a = b$. The following lemma is obvious.

Lemma 1. *Let θ be a relation in a semigroup S defined as follows. $a \, \theta \, b$ if $xa = xb$ for each $x \in S$. Then θ is a congruence relation on S.*

Given a congruence relation η on S and $x \in S$, $[x] \, \eta$ will denote the block of η containing x.

Remark. The quotient semigroup S/θ (θ is the congruence relation of Lemma 1) need not be left reductive as the following example shows. $S = (\{0, 2, 4, 8\}, .)$, where . denotes multiplication mod 16. S/θ consists of the θ-blocks $\bar{0} = \{0,8\}$, $\bar{2} = \{2\}$, $\bar{4} = \{4\}$. $x . \bar{0} = x . \bar{4}$ holds for each $x \in S/\theta$.

Denote $\theta(S)$ the congruence relation θ (on S) of Lemma 1. We define congruence relations θ_n ($n = 1, 2, \ldots$) on S by induction: $\theta_1 = \theta(S)$, and if θ_n is given, θ_{n+1} is the congruence relation on S, induced by the congruence relation $\theta(S/\theta_n)$. Obviously $\theta_i \leqq \theta_{i+1}$ for all $i \in \{1, 2, \ldots\}$. The congruence relation $\bigvee_{i=1}^{\infty} \theta_i$ will be denoted by θ^*. The congruence relations θ_n and θ^* are characterized in Lemma 2 and Theorem 1.

Given a natural number n, let $S^n = \{a_1 a_2 \ldots a_n \mid a_i \in S\}$.

Lemma 2. *$a \, \theta_n b$ iff $xa = xb$ for all $x \in S^n$.*

Proof. The assertion is true for $n = 1$. Suppose the assertion true for $n - 1$. Given $x \in S$, denote $\bar{x} = [x] \, \theta_{n-1}$. Then the following assertions are equivalent: $a\theta_n b$, $\bar{a}\theta \, (S/\theta_{n-1}) \, \bar{b}$, $\bar{x}\bar{a} = \bar{x}\bar{b}$ for all $x \in S$, $xa \; \theta_{n-1} \; xb$ for all $x \in S$, $txa = txb$ for all $x \in S$ and all $t \in S^{n-1}$, $sa = sb$ for all $s \in S^n$.

Theorem 1. *$a \, \theta^* \, b$ iff $n \in N$ exists such that $xa = xb$ for all $x \in S^n$. S/θ^* is a left reductive semigroup and θ^* is the least element in the set of all congruence relations η on S such that S/η is left reductive.*

Proof. $a \, \theta^* \, b$ iff $a \, \theta_n \, b$ for some $n \in N$, whence the first assertion follows. $\bar{x}\bar{a} = \bar{x}\bar{b}$ for all $\bar{x} \in S/\theta^*$ imply $xa \, \theta^* \, xb$ for all $x \in S$, hence $txa = txb$ for all $x \in S$ and all $t \in S^k$ for some $k \in N$, i.e. $ya = yb$ for all $y \in S^{k+1}$, hence $a \, \theta^* \, b$, i.e. $\bar{a} = \bar{b}$. Thus S/θ^* is left reductive. Let η be a congruence relation on S such that S/η is left reductive. Given $x \in S$, let $\tilde{x} = [x]\eta$. We prove $\theta_n \leqq \eta$ for all $n \in N$ by induction. $a\ \theta_1\ b$ implies $xa = xb$ for all $x \in S$, hence $\tilde{x}\tilde{a} = \tilde{x}\tilde{b}$ for all $\tilde{x} \in S/\eta$, hence $\tilde{a} = \tilde{b}$, i. e. $a\ \eta b$. Thus $\theta_1 \leqq \eta$. Suppose $\theta_{n-1} \leqq \eta\ (n > 1)$. If $a\theta_n b$ then $xa\ \theta_{n-1}\ xb$ for all $x \in S$, hence $xa\eta xb$ for all $x \in S$, hence $a\eta b$. It follows $\theta_n \leqq \eta$. Hence $\theta^* \leqq \eta$.

2. THE CONSTRUCTION

Theorem 2. (a) *Let S be a semigroup and let with each $x \in S$ a set $T_x \neq \emptyset$ be associated, and let $T_x \cap T_y = \emptyset$ for all $x \neq y$. Let a mapping $f_x^y : T_x \to T_y$ be given for all couples $x, y \in S$ such that $y = ux$ for some $u \in S$. Suppose further $f_y^z \circ f_x^y = f_x^z$. Given $a, b \in \bigcup_{x \in S} T_x = T$, set $a \circ b = f_x^{xy}(a)$, where $a \in T_x$, $b \in T_y$. Then $(T, \circ)$ is a semigroup and each set T_x is contained in a block of the congruence relation θ of Lemma 1. If S is left reductive then the blocks of θ are exactly the sets T_x $(x \in S)$.*

(b) *Let θ be the congruence relation of Lemma* 1 *on a semigroup T. Then $ab = ac$ for all $a \in [x]\ \theta$ and all $b, c \in [y]\ \theta$. Hence $f_{[x]\theta}^{[xy]\theta}$: $a \mapsto ab$ $(b \in [y]\ \theta)$ is a mapping on $[x]\ \theta$ to $[xy]\ \theta$ and for all c, $d \in T$, $cd = f_{[c]\theta}^{[cd]\theta}$* (c).

Remark. The semigroup $(T, \circ)$ in (a) will be said to be derived from the semigroup S by a θ-construction. Thus the assertion (b) says that any semigroup T can be derived from T/θ by a θ-construction.

Proof. (a) The associativity of the operation $\circ$ is obvious. If a, b belong to the same set T_x and $c \in T_y$ then $c \circ a = f_y^{yx}(c) = c \circ b$. Hence $a\ \theta\ b$. Let S be left reductive and let $a \in T_x$, $b \in T_y$, $x \neq y$. Then $z \in S$ exists such that $zx \neq zy$. If $c \in T_z$ then $c \circ a = f_z^{zx}(c) \neq f_z^{zy}(c) = c \circ b$, hence $a\ \theta\ b$ does not hold.

(b) $b, c \in [y]\ \theta$ implies $b\ \theta\ c$, hence $ab = ac$ for each $a \in T$. The remaining assertions are obvious.

Theorem 3. *Let S be a semigroup whose congruence relations satisfy the ACC. Then semigroups $S_0, S_1, \ldots, S_n = S$ exist such that S_0 is left reductive and each S_{i+1} can be derived from S_i by a θ-construction.*

Proof. From the ACC it follows that $\theta^* = \theta_n$ for some $n \in N$. Let $S_n = S$, $S_{n-1} = S/\theta$, $S_{n-2} = S_{n-1}/\theta$, $\ldots$, $S_0 = S_1/\theta$. Then $S_0 \cong S/\theta_n$ is left reductive by Theorem 1 and, by Theorem 2 (b), S_k can be derived from S_{k-1} by a θ-construction for each $k = 1, \ldots, n$.

REFERENCES

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M. Kolibiar
Department of Mathematics
Šmeralova 2b, Bratislava
Czechoslovakia