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EXTENSIONS OF MAPPINGS OF FINITE BOOLEAN ALGEBRAS TO HOMOMORPHISMS

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This paper is a continuation of the paper [1], in which a matrix representation of homomorphic mappings in Boolean algebra has been given. It is obvious that investigations of homomorphisms are essentially simplified by this representation. As in paper [1] also here, I do not deal with an abstract Boolean algebra, but with its isomorphic representation — the B-modul (see [2]).

Let $\mathfrak{M}_m$ be the m-dimensional B-modul, i. e. Boolean algebra with 2^m elements and $\mathfrak{M}_n$ the n-dimensional B-modul, i. e. Boolean algebra with 2^n elements. Let us denote by φ_i the B-vectors from $\mathfrak{M}_m$, by ψ_i the B-vectors from $\mathfrak{M}_n$. We denote o the zero vector in both B-moduls, since there is no danger of misunderstanding.

By the matrix representaion it is possible to solve the following problem:

Let there be given mapping of a subset of $\mathfrak{M}_m$ into $\mathfrak{M}_n$ by relations:

$$\text{(S)}\qquad \begin{array}{cccc} \varphi_i \in \mathfrak{M}_m, & \psi_i \in \mathfrak{M}_n, & \varphi_i \to \psi_i & i = 1, 2, \ldots, k < 2^m \\ & o \in \mathfrak{M}_m & o \to o \in \mathfrak{M}_n & \end{array}$$

We have to determine the homomorphic mappings α of $\mathfrak{M}_m$ into $\mathfrak{M}_n$ which fulfil the relations (S).

This problem is the so called problem of extension of mappings to homomorphisms. We shall show bellow that this problem can have no, one or more solutions. Let the relations (S) be prescribed; for the sake of brevity let us introduce the following notation:

$$\varphi_i = (f_1^{(i)}, f_2^{(i)}, \ldots, f_m^{(i)}), \qquad \psi_i = (g_1^{(i)}, g_2^{(i)}, \ldots, g_n^{(i)}), \qquad i = 1, 2, \ldots, k.$$

Now there holds the following theorem:

Theorem 1. Let A be a B-matrix of the type m/n, representing a homomorphic mapping α of $\mathfrak{M}_m$ into $\mathfrak{M}_n$, this mapping fulfilling conditions (S).

$$\text{If } (f_p^{(1)}, f_p^{(2)}, \ldots, f_p^{(k)}) \neq (g_q^{(1)}, g_q^{(2)} \ldots, g_q^{(k)}),$$

then $a_{pq} = 0$ (a_{pq} is an element of the matrix A in therow and q-th column).

Proof: Let the assumption of the theorem hold and $a_{pq} = 1$. Then the p-th row of B-matrix A is a B-vector

$$a_p = (a_{p1}, \ldots, a_{p,\,q-1}, 1, a_{p,q+1}, \ldots, a_{pn});$$

for the p-th vector of the base of modul $\mathfrak{M}_m$, i.e. for $e^{(p)}$ holding $\alpha(e^{(p)}) = a_p$. Of course, by the definition 5 (see [1]), there holds $\psi_1 = f_1^{(1)}a_1 + \ldots + f_{p-1}^{(1)}a_{p-1} + f_p^{(1)}a_p + \ldots + f_m^{(1)}a_m$.

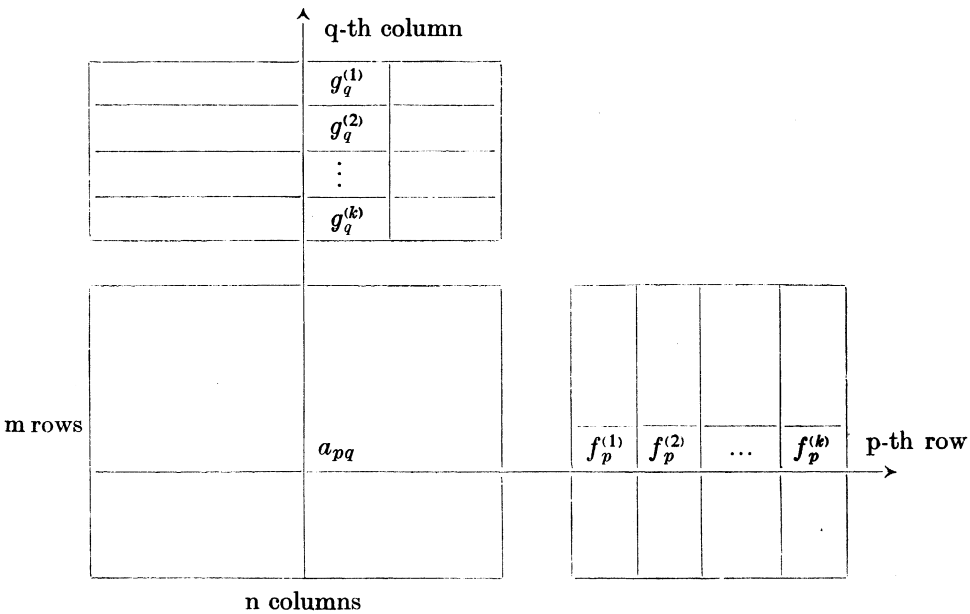
By the assumption, the q-th coordinate of the B-vector a_p is equal to *1*. But there can be at most one unity in each column of the matrix A (see theorem 1 in [1]). Thus the vector a_p has a unity only in the q-th column. Then the q-th coordinate of the vector ψ_1 is equal to $f_p^{(1)} . 1 = f_p^{(1)}$ and from this $g_q^{(1)} = f_p^{(1)}$. Analogously for all i, thus

$$(f_p^{(1)}, \dots, f_p^{(k)}) = (g_q^{(1)}, \dots, g_q^{(k)})$$

which is a contradiction.

This theorem implies a simple algorithm:

We shall construct a matrix with m rows and n columns where all $m . n$ places are still not filled in. Behind this matrix, we shall write vertically vectors $\varphi_1, \varphi_2, \dots, \varphi_k$, over this matrix, we shall write horizontally $\psi_1, \psi_2, \dots, \psi_k$, see diagram:



Let us write the prescribed diagram and fill the coordinates of B-vectors φ_i, ψ_i. Now we compare the sequences of 0 and 1 in the vectors (in the first diagram in frames). If these sequences are different, we fill 0 as an element in the crossing of this row and column. In the opposite case we can fill 1 of course, in each column at most one. By this way we receive all desired matrices. It is possible to fill 1 in all places a_{pq}, where

$$(f_p^{(1)}, \dots, f_p^{(k)}) = (g_q^{(1)}, \dots, g_q^{(h)}).$$

The matix C derived in this way will be called *the decomposition matrix*. The set of all matrices representing all desired homomorphic mappings form the decomposition if this matrix C (see definition 4 in [1]). Thus number of all possible homomorphisms is the number of matrices in the decomposition of the matrix C. It is the product of the sums of unities in the non zero columns of the matrix C.

It can happen that the images of vestors φ_i derived from the matrix C (constructed in usual way) are not ψ_i as given by the prescription (S). In this case the homomorphic mapping with the desired property does not exist (as it is easy to show).

This algorithm solves the problem of existence, number and construction of homomorphic extension of the given mapping of a finite Boolean algebra into another Boolean algebra.

Example 1. Find the homomorphic mappings of $\mathfrak{M}_5$ into $\mathfrak{M}_4$ fulfilling the relations:

$$\begin{aligned} \varphi_1 &= (0\ 1\ 1\ 0\ 1) \to \psi_1 = (1\ 0\ 1\ 1) \\ \varphi_2 &= (0\ 1\ 0\ 1\ 0) \to \psi_2 = (0\ 0\ 1\ 0) \\ o &= (0\ 0\ 0\ 0\ 0) \to \ o = (0\ 0\ 0\ 0) \end{aligned}$$

Then the prescribed diagram is:

		φ_1	φ_2
$\psi_1 =$	(1 0 1 1)		
$\psi_2 =$	(0 0 1 0)		
		0	0
		1	1
		1	0
		0	1
		1	0

Now we fill it by the given algorithm:

$$\begin{array}{rc|c|c} & \begin{matrix}1\,0\,1\,1\\0\,0\,1\,0\end{matrix} & & \\ \hline C = & \begin{bmatrix}0\,1\,0\,0\\0\,0\,1\,0\\1\,0\,0\,1\\0\,0\,0\,0\\1\,0\,0\,1\end{bmatrix} & \begin{matrix}0\\1\\1\\0\\1\end{matrix} & \begin{matrix}0\\1\\0\\1\\0\end{matrix} \end{array}$$

$$s = 2\,.\,1\,.\,1\,.\,2 = 4$$

Thus there are just 4 matrices representing 4 possible homomorphic mappings. These matrices form the decomposition of the matrix C:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The image of each B-vector from $\mathfrak{M}_5$ will be found by means of matrices A_i.

The mappings constructed in example 1 are evidently mappings which map 1 onto 1, which is easy to prove. Now in the following example we shall show the form of a matrix representing mapping without this property.

Example 2. Find homomorphisms of $\mathfrak{M}_6$ into $\mathfrak{M}_4$ fulfilling relations:

$$(S_1) \qquad \begin{aligned} \varphi_1 &= (1\ 1\ 0\ 0\ 0\ 0) \to (0\ 1\ 1\ 0) = \psi_1 \\ \varphi_2 &= (0\ 0\ 1\ 0\ 0\ 1) \to (0\ 0\ 1\ 0) = \psi_2 \\ o &= (0\ 0\ 0\ 0\ 0\ 0) \to (0\ 0\ 0\ 0) = o \end{aligned}$$

Write directly the diagram:

$$
\begin{array}{cc|c|c}
 & \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} & & \\
\hline
C = & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{matrix} \\
\hline
\end{array}
$$

From the decomposition of the matrix C we get 8 different matrices representing the homomorphic mappings. The third column of each of these matrices is a zero column, hence the mappings are necessarily of the type "into", and the desired homomorphic mapping fulfilling (S_1) does not exist because vectors ψ_1, ψ_2 have not preimages in each possible homomorphisms.

It is easy to prove the theorem:

Theorem 2. The matrix A of type m/n which has in h columns only 0 ($1 \leqq h \leqq n$) and in all remaining $n - h$ columns just one unity and k_j unities in the j-th row represents the homomorphic mapping of $\mathfrak{M}_m$ into $\mathfrak{M}_n$. Let $r_j = \max(1, k_j) - 1$, $r = \sum_{j=1}^{n} r_j$. Then there exist just $2^n - 2^{n-h-r}$ vectors in the $\mathfrak{M}_n$, which have not a preimage in $\mathfrak{M}_m$.

It is evident that by the matrix representation it is easy to determine whether the given homomorphism is of the type "onto" or "into". We can prove the theorem:

Theorem 3. Given the relations (S). If the decomposition matrix C has at most one unity in each row, then only one of the following three alternatives is holding:

1. There does not exist a homomorphic mapping of $\mathfrak{M}_m$ into $\mathfrak{M}_n$ fulfilling (S).
2. There exists a homomorphic mapping of $\mathfrak{M}_m$ *into* $\mathfrak{M}_n$ fulfilling (S), but there does not exist a homomorphic mapping of the type "onto" fulfilling (S).
3. There exists a homomorphic mapping $\mathfrak{M}_m$ *onto* $\mathfrak{M}_n$ fulfilling (S), but there does not exist a homomorphic mapping of the type "into" fullfilling (S) which is not of the type "onto".

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