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ON SUBSTITUTION OF OPERATIONS IN SYSTEMS OF EQUATIONS OVER ALGEBRAS

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The mapping transforming a system of equations over an algebra into another algebra with different operations (from a different class of algebras) is frequently constructed for solving this system (for instance in operator calculus). A solution of the system in the second algebra is transformed back again into the first algebra and in individual cases it is proved that the transformed back solution is a solution of the initial system. The theorem which gives conditions for back transformation can be generalized and assumptions of it can be weakened.

The conceptions of the systems of equations over algebra and of the regularizer are taken from [1] and [2].

1.

By the symbol $\mathfrak{A} = (A, O_\Gamma)$ there is denoted the algebra with the set of generators A and the set of operations O_Γ. For each operation $o_\gamma \in O_\Gamma$ there exists an ordinal number k_γ — the so called arity of o_γ. By the symbol $\{a_\alpha, \alpha < k_\gamma\}$ it is denoted a sequence of the type k_γ formed by the elements a_α. Let $a_\alpha \in \mathfrak{A}$ for $\alpha < k$. The result of operation o_γ for elements $\{a_\alpha, \alpha < k_\gamma\}$ is denoted by $o_\gamma(a_\alpha, k_\gamma)$

Let $\mathscr{A}_X$ be the set of all expressions consisting of elements of the algebra $\mathfrak{A} = (A, O_\Gamma)$, of the O_Γ and of the set $X = \{x_\mu, \mu < s\}$ (where $X \cap A = \emptyset$, $X \cap O_\Gamma = \emptyset$) which would give elements of $\mathfrak{A}$ if the elements of X were replaced by elements of $\mathfrak{A}$; that is, expressions with the right number of elements (of $\mathfrak{A}$ or X) after each operation-symbol.

Let us introduce the equivalence ω on $\mathscr{A}_X$: an element $\tau \in \mathscr{A}_X$ is equivalent to element $\vartheta \in \mathscr{A}_X$, symbolically $\tau\, \omega\, \vartheta$ iff $\tau = \vartheta$ for each replacing elements of X by elements of $\mathfrak{A}$. We can introduce an operation $o_\gamma \in O_\Gamma$ for elements of $\mathscr{A}_X$:

$\tau\, \omega\, o_\gamma(\vartheta_\alpha, k_\gamma)$, $\vartheta_\alpha \in \mathscr{A}_X$ iff $\bar{\tau} = o_\gamma(\bar{\vartheta}_\alpha, k_\gamma)$ for arbitrary replacing elements of X by lements of $\mathfrak{A}$ (where $\bar{\vartheta}_\alpha$ is obtained from ϑ_α by replacing elements of X by elements of $\mathfrak{A}$).

It is clear that ω is a congruence relation on $\mathscr{A}_X$.

Definition 1. The factor algebra $\mathscr{A}_X/\omega$ is called *formal $\mathfrak{A}$-polynomial algebra* and is denoted by For $(\mathfrak{A}, X)$. Each element of For $(\mathfrak{A}, X)$ is called the *$\mathfrak{A}$-term* (or briefly the *term*).

Any term V of For $(\mathfrak{A}, X)$ generated only by the set $\{x_\mu, \mu < k\} \cup A$ and by operations O'_Γ, where $k \leqq s$, $O'_\Gamma \subseteq O_\Gamma$ is denoted by $V(x_\mu, k, O'_\Gamma)$.

Remark. $\mathfrak{A}$ is a subalgebra of For $(\mathfrak{A}, X)$ because $Y \subseteq X \Rightarrow$ For $(\mathfrak{A}, Y) \subseteq$ For $(\mathfrak{A}, X)$ and $\mathfrak{A} =$ For $(\mathfrak{A}, \emptyset)$.

Let $\mathfrak{A} = (A, O_\Gamma)$ be an algebra of the class Λ, $\mathfrak{B} = (B, O_H)$ be an algebra of the of the class Λ'.

Definition 2. The mapping φ of For $(\mathfrak{A}, X)$ into For $(\mathfrak{B}, X)$ is said to be the *S-mapping* iff the following conditions hold:

(i) the image of the $\mathfrak{A}$-term $o_\gamma(x_\mu, k_\gamma)$ is the $\mathfrak{B}$-term

$$V_\gamma(x_\mu, k_\gamma, O_H)$$

(ii) $\varphi|\mathfrak{A}$ is the mapping of $\mathfrak{A}$ into $\mathfrak{B}$

(iii) for each sequence $\{a_\mu, \mu < k\}$, $a_\mu \in \mathfrak{A}$ the identity $\varphi(o_\gamma(a_\mu, k_\gamma)) = V_\gamma(\varphi(a_\mu), k_\gamma, O_H)$ holds.

Then the operation $o_\gamma \in O_\Gamma$ is called *substitutable by the operation O_H* in the algebra $\mathfrak{B}$.

Remark. A special case of S-mapping is a homomorphic mapping (i.e. $\mathfrak{A}, \mathfrak{B} \in \Lambda$, $O_\Gamma = O_H$, $V_\gamma(x_\mu, k_\gamma, O'_H) = o_\gamma(x_\mu, k_\gamma)$, $o_\gamma \in O_H$). If $\mathfrak{A}$ is a grupoid and $\mathfrak{B}$ is an algebrá with one binary operation and two suitable defined unary operations, then the conception of substitutability can be equal to the conception of isotopy.

Theorem 1. *The equivalence relation Φ induced by an S-mapping on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$.*

Proof. Let $\{a_\mu, \mu < k\}, \{b_\mu, \mu < k\}$ be sequences of elements of $\mathfrak{A}$, φ be an S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$ and $\langle a_\mu, b_\mu\rangle \in \Phi$ for each $\mu < k$, i.e. $\varphi(a_\mu) = \varphi(b_\mu)$. Then $\varphi(o_\gamma(a_\mu, k_\gamma)) = V_\gamma(\varphi(a_\mu) . k_\gamma O'_H) = V_\gamma(\varphi(b_\mu), k_\gamma, O'_H) = \varphi(o_\gamma(b_\mu, k_\gamma))$, thus $\langle o_\gamma(a_\mu, k_\gamma), o_\gamma(b_\mu, k_\gamma)\rangle \in \Phi$ for each k_γ-ary operation $o_\gamma \in O_\Gamma$. Accordingly, Φ is a congruence relation on $\mathfrak{A}$.

EXAMPLES ON THE SUBSTITUTION OF OPERATIONS

1. Let $\mathfrak{A}$ be the Boolean algebra with n generators $\{a_\mu, \mu = 0, 1, \ldots, n-1\}$, $\mathfrak{B}$ be the Boolean ring with unity generated by the set of generators $\{b_\mu, \mu = 0, 1, \ldots, n-1\}$. Let φ be the mapping of For $(\mathfrak{A}, X)$ into For $(\mathfrak{B}, X)$ for which $\varphi(a_\mu) = b_\mu$ for $\mu < n$, and for arbitrary $x_0, x_1 \in X$ is

$$\varphi(x_0 \cup x_1) = \varphi(x_0) + \varphi(x_1) - \varphi(x_0) \cdot \varphi(x_1)$$

$$\varphi(x_0 \cap x_1) = \varphi(x_0) \cdot \varphi(x_1)$$

$$\varphi(\bar{x}_0) = 1 - \varphi(x_0).$$

Then φ is the S-mapping and operation, $\cup$ is substitutable in $\mathfrak{B}$ by $O_H = \{+, -, .\}$. Analogously for other Boolean operations. The inverse mapping φ^{-1} is the S-mapping of $\mathfrak{B}$ into $\mathfrak{A}$. It follows directly from [8] and [3]. Then for example the operation $+$ of $\mathfrak{B}$ is substitutable in $\mathfrak{A}$ by the set $O_\Gamma = \{\cup, \cap, \bar{\ }\}$ of all operations of $\mathfrak{A}$ because

$$\varphi^{-1}(x_0 + x_1) = (\varphi^{-1}(x_0) \cap \overline{\varphi^{-1}(x_1)}) \cup (\overline{\varphi^{-1}(x_0)} \cap \varphi^{-1}(x_1)).$$

2. Let $\mathfrak{A}$ be the set of non-zero complex functions of the real variable t which have the continuous first derivative in the interval $\langle 0, \infty)$ and fulfil $|f(t)| \leqq Me^{St}$, where $M \geqq 0$, $S \geqq 0$ are constants, let the set of operations of $\mathfrak{A}$ be $O_\Gamma = \{+, -, ., :,$ derivative, integral$\}$. Let $\mathfrak{B}$ be the field of operators, where $O_H = \{+, -,$ operator product, operator quotient, multiplications by a constants$\}$. Then there exist various

S-mappings of $\mathfrak{A}$ into $\mathfrak{B}$, for example the Laplace transformation, the Garson-Laplace transformation, the Fourier transformation etc. Operations of $\mathfrak{A}$ are substitutable in $\mathfrak{B}$ by the operations of $\mathfrak{B}$.

2.

Let $\mathscr{A}$ be the set of elements of For $(\mathfrak{A}, X)$, $X = \{x_\mu, \mu < s\}$ and $\mathfrak{A} = (A, O_\Gamma)$.

Definition 3. The subset E of the Cartesian product $\mathscr{A} \times \mathscr{A}$ is said to be the *system of equations over* $\mathfrak{A}$, each pair $\langle \tau, \vartheta \rangle \in E$ of $\mathfrak{A}$-terms τ, ϑ is called the *equation*. Elements of X (resp. of A) generating $\mathfrak{A}$-terms τ, ϑ are called *unknowns* (resp. *parameters*) of the equation $\langle \tau, \vartheta \rangle \in E$.

Definition 4. A homomorphic mapping h of For $(\mathfrak{A}, X)$ into $\mathfrak{A}' = (A', O_\Gamma)$, where $\mathfrak{A}'$, $\mathfrak{A}$, For $(\mathfrak{A}, X)$ are of the same class of algebras, is called the *characteristic mapping* of the system E iff $h(\tau) = h(\vartheta)$ for each $\langle \tau, \vartheta \rangle \in E$ and $h(\mathrm{For}(\mathfrak{A}, X)) = h(\mathfrak{A})$. If $h \mid \mathfrak{A}$ is an isomorphic mapping of $\mathfrak{A}$ into $\mathfrak{A}'$, the characteristic mapping h is said. to be *proper*. The congruence relation induced by h on For $(\mathfrak{A}, X)$ is called the *regularizer* of the system E. If h is proper, the regularizer induced by h is called *proper*.

By the symbol $\bar{\tau}$ we denote the $\mathfrak{A}$-term τ where all elements of X which generate τ are replaced by elements of $\mathfrak{A}$ and each x_μ is replaced by the same $a_\mu \in \mathfrak{A}$ in all places in τ.

Definition 5. Let E be the system of equations over $\mathfrak{A}$ and $\sim$ be a regularizer of E. The sequence $\{V_\mu(a_\alpha, k_\mu, O_\Gamma), \mu < s\}$, where $a_\alpha \in \mathfrak{A}$, is said to be the *solution of the system* E with the regularizer $\sim$ iff we obtain $\langle \bar{\tau}, \bar{\vartheta} \rangle \in \sim$ for each $\langle \tau, \vartheta \rangle \in E$ by replacing each element x_μ by the element $V_\mu(a_\alpha, k_\mu, O_\Gamma)$. If the regularizer $\sim$ is proper, the solution is called *proper*.

In [2] it is shown that the solution $\{V_\mu(a_\alpha, k_\mu, O_\Gamma), \mu < s\}$ is proper iff $\bar{\tau} = \bar{\vartheta}$ for each $\langle \tau, \vartheta \rangle \in E$. Accordingly, the proper solution is the solution in sense of the classical definition (see for example [6]). The definition 5 is, however, more general than that one.

3.

Let E be the system of equations over an algebra $\mathfrak{A} = (A, O_\Gamma)$, let φ be an S-mapping of $\mathfrak{A}$ into $\mathfrak{B} = (B, O_H)$. The mapping φ maps each $\mathfrak{A}$-term τ onto $\mathfrak{B}$-term $\varphi(\tau)$ and each equation $\langle \tau, \vartheta \rangle$ of E onto $\mathfrak{B}$-equation $\langle \varphi(\tau), \varphi(\vartheta) \rangle$. Let us denote by $\varphi(E)$ the set of all $\langle \varphi(\tau), \varphi(\vartheta) \rangle$ for $\langle \tau, \vartheta \rangle \in E$.

We use frequently (for example in the operator calculus) the theorem on transforming of the solution of $\varphi(E)$ onto a solution of E. This theorem can be generalized for arbitrary algebras:

Theorem 2. *Let E be a system of equations over $\mathfrak{A}$, φ be an injective S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$ and ψ be an injective S-mapping of $\mathfrak{B}$ into $\mathfrak{A}$. If $\{V_\mu(b_\alpha, k_\mu O_H), \mu < s\}$ is a solution of the system $\varphi(E)$ with regularizer $\sim$, then*

$$\{\psi(V_\mu(b_\alpha, k_\mu, O_H)), \mu < s\}$$

is a solution of the system E with regularizer $\sim_\varphi$ *defined by the rule:*

$$\langle a, b\rangle \in \sim_\varphi \text{ iff } \langle \varphi(a), \varphi(b)\rangle \in \sim \tag{P}$$

We can however, weaken the assumption of this theorem and extend the range of applications of it. The assumption of existence of „inverse" S-mapping ψ is too strong for the applications where the theorem can bring new results (for new transformations in the operator calculus, for modeling various systems etc.).

Lemma. *Let* $\mathfrak{A} = (A, O_\Gamma)$, φ *be an S-mapping of* $\mathfrak{A}$ *into* $\mathfrak{B} = (B, O_H)$. *If* $\sim$ *is a congruence relation on* $\mathfrak{B}$, *the relation* $\sim$a *defined by* (P) *is the congruence relation on* $\mathfrak{A}$.

Proof. If is evident that $\sim_\varphi$ is an equivalence relation on $\mathfrak{A}$ because (P) implies the reflexivity, transitivity and symmetry of relation $\sim_\varphi$ for congruence $\sim$. Let $\sim_\varphi$ be not a congruence relation. Then there exists at least one sequence $\{\langle a_\mu, b_\mu\rangle, \mu < k\}$ and at least one k_γ-ary operation $o_\gamma \in O_\Gamma$ so that $\langle a_\mu, b_\mu \rangle \in \sim_\varphi$ for $\mu < k$ and $\langle o_\gamma(a_\mu, k_\gamma), o_\gamma(b_\mu, k_\gamma)\rangle \notin \sim_\varphi$. From it follows by (P):

$$\langle \varphi(o_\gamma(a_\mu, k_\gamma)), \varphi(o_\gamma(b_\mu, k_\gamma)) \rangle \notin \sim$$

which is a contradiction because $\sim$ is a congruence relation by the asumption and $\langle \varphi(o_\gamma(a_\mu, k_\gamma)), \varphi(o_\gamma(b_\mu, k_\gamma))\rangle = \langle V_\gamma(\varphi(a_\mu), k_\gamma, O'_H), V_\gamma(\varphi(b_\mu), k_\gamma, O'_H)\rangle \in \sim$.

Theorem 3. *Let* $\mathfrak{A} = (A, O_\Gamma)$, $\mathfrak{B} = (B, O_H)$, φ *be a S-mapping of* $\mathfrak{A}$ *into* $\mathfrak{B}$. *Let E be a system of equations over* $\mathfrak{A}$. *If* $\{V_\mu(b_\alpha, k_\mu, O'_H), \mu < s\}$ *is a solution of the system* $\varphi(E)$ *with regularizer* $\sim$ *and* W_μ *is an arbitrary element of* $\mathfrak{A}$ *fulfilling* $\varphi(W_\mu) = V_\mu$ *for each* $\mu < s$, *then* $\{W_\mu, \mu < s\}$ *is a solution of E with the regularizer* $\sim_\varphi$ *given by* (P).

Proof. By the lemma $\sim_\varphi$ is a congruence relation on $\mathfrak{A}$. Let $\bar{\tau}$ be an $\mathfrak{A}$-term τ where each x_μ is replaced by $W_\mu \in \mathfrak{A}$, $\overline{\varphi(\tau)}$ be $\mathfrak{B}$-term $\varphi(\tau)$, where x_μ is replaced by $V_\mu \in \mathfrak{A}$ and let $\varphi(W_\mu) = V_\mu$, where $\{V_\mu, \mu < s\}$ is a solution of $\varphi(E)$.

By the condition (iii) od the definition 2 we obtain $\varphi(\bar{o}_\gamma) = \overline{\varphi(o_\gamma)}$, where $o_\gamma(x_\mu, k_\gamma)$ is an $\mathfrak{A}$-term. By the theorem 1 we obtain $\varphi(\bar{\tau}) = \overline{\varphi(\tau)}$ for an arbitrary $\mathfrak{A}$-term. Thus $\overline{\langle\varphi(\tau), \varphi(\vartheta)\rangle} \in \sim$ implies $\langle \varphi(\bar{\tau}), \varphi(\bar{\vartheta})\rangle \in \sim$ and by (P) we obtain $\langle \bar{\tau}, \bar{\vartheta}\rangle \in \sim_\varphi$. Accordingly, $\{W_\mu, \mu < s\}$ is really the solution of E with the regularizer $\sim_\varphi$.

The theorem 2 now follows from the theorem 3.

4.

It is possible that the system E has other solutions which can not be obtained from solu tions of $\varphi(E)$ hy the theorem 3. The solution of E obtained from the solution $\{V_\mu, \mu < s\}$ of $\varphi(E)$ by this theorem is called *induced by the solution* $\{V_\mu, \mu < s\}$. The solution of E induced by proper solution of $\varphi(E)$ need not be proper.

Theorem 4. *Let E be a system of equations over* $\mathfrak{A}$, φ *be an S-mapping of* $\mathfrak{A}$ *into* $\mathfrak{B}$. *The solutio n of E induced by the solution* $\{V_\mu, \mu < s\}$ *of* $\varphi(E)$ *is proper iff:*

(1) $\{V_\mu, \mu < s\}$ *is a proper solution of* $\varphi(E)$

(2) $\varphi^{-1}[b] \cap \mathfrak{A}$ *is a one-element set for each* $b \in \mathfrak{B}$.

Proof. T he sufficiency is evident. Necessity: Let $\{V_\mu, \mu < s\}$ be not proper solution of $\varphi(E)$, i.e. $\sim \not\equiv =$ and let the induced solution be proper. i.e. $\sim_\varphi \equiv =$.

Then from $\langle \bar{\tau}, \bar{\vartheta} \rangle \in \sim_\varphi$ we have $\bar{\tau} = \bar{\vartheta}$ and by (P) we obtain $\varphi(\bar{\tau}) = \varphi(\bar{\vartheta})$ for each $\langle \varphi(\tau), \varphi(\vartheta) \rangle \in \varphi(E)$. From it $\sim \equiv =$ and thus $\{V_\mu, \mu < s\}$ is proper which is a contradiction. Let $\varphi^{-1}[b] \cap \mathfrak{A}$ be not one-element set and $\sim \equiv =$, then $\varphi(\bar{\tau}) = \varphi(\bar{\vartheta})$. By the theorem 1, $\sim_\varphi$ is a congruence relation, but $\varphi^{-1}[b] \cap \mathfrak{A}$ is not one-element set, i.e. $\sim_\varphi \not\equiv =$ which is a contradiction with the assumptions of the proof again.

For applications the following sufficiency condition can be often use:

Corollary 5. *Let φ be an injective S-mapping of $\mathfrak{A}$ into $\mathfrak{B}$, E be a system of equations over $\mathfrak{A}$. Then the solution of E induced by a proper solution of $\varphi(E)$ is proper.*

Proof. The assumptions of corollary fulfils (1) of the theorem 4. From injectivity of φ we obtain condition (2) of the theorem 4. By this theorem we obtain the assertion of corollary.

Theorem 3 and corollary 5 can be applied to the operator calculus if the transformation into the field of operators is injective but there does not exist substitutability for each operator operation into initial functional algebra (i.e. the inverse mapping of the S-mappint φ of $\mathfrak{A}$ into $\mathfrak{B}$ is not an S-mapping of $\mathfrak{B}$ into $\mathfrak{A}$ for each operation of $\mathfrak{B}$).

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