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AN OSCILLATION CRITERION FOR THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS

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We investigate a linear differential equation of the third order of the form

$$y''' + p(t)\,y' + q(t)\,y = 0. \tag{L}$$

We assume that the functions $p(t)$, $q(t)$ are continuous and do not change sign on $[a, \infty)$.

This equation (L) was studied by several authors, namely Greguš, Hanan [1], Ráb, Švec, Zlámal [4], and the main results have been collected by Lazer [2] giving the most important papers of the above mentioned authors in the list of references. Some new results were obtained by Singh [3].

Let $p(t) \in C^1[a, \infty)$. Then investigating this equation (L), Mammana's identity written in the form

$$F(y(t)) = F(y(a)) + \int_a^t [2q(s) - p'(s)]\, y^2(s)\, \mathrm{d}s, \tag{M}$$

where $F(y(t)) = y'^2(t) - 2y(t)\,y''(t) - p(t)\,y^2(t)$
has a very important role.

A nontrivial solution of the equation (L) is called oscillatory if it has infinitely many zeros on $[a, \infty)$, otherwise nonoscillatory.

In the proofs of some theorems in the papers [2], [3] there is used the procedure given in the form of the following.

Lemma 1. *Let $u_i(t) \in C^r[a, \infty)$ be functions, c_{in} constants, $n >$ a positive integers, $i = 1, 2, .., s$. Let the sequences $\{y_n^{(z)}\}$ be defined by the relations*

$$y_n^{(z)} = \sum_{i=1}^{s} c_{in} u_i^{(z)}, \qquad \sum_{i=1}^{s} c_{in}^2 = 1, \qquad z = 0, 1, \dots, m \leqq r.$$

Then there exists the sequence $\{n_j\}$ such that $c_{in_j} \to c_i$ and $\{y_{n_j}^{(z)}\}$ converge on every *finite subinterval of* $[a, \infty)$ *uniformly to the functions*

$$y^{(z)} = \sum_{i=1}^{s} c_i u_i^{(z)}, \qquad \sum_{i=1}^{s} c_i^2 = 1 \quad \text{for} \quad n_j \to \infty.$$

We shall consider the case of $p(t) \geqq 0$, $q(t) < 0$.

Lemma 2. *Let* $p(t) \geqq 0$, $q(t) < 0$ *and* $y(t)$ *be a nontrivial solution of the equation* (L) *satisfying* $y(t)\,y'(t) \neq 0$ *on* $[a, \infty)$. *Then* $y(t)\,y'(t) > 0$ *holds on this interval.*

Proof: Let $y(t)\,y'(t) < 0$. We can suppose without loss of generality that $y(t) > 0$. Then on $[a, \infty)$ there holds

$$-y'''(t) = p(t)\,y'(t) + q(t)\,y(t) < 0.$$

The function $y''(t)$ is increasing and $b \geqq a$ exists such that on $[b, \infty)$ there holds either $y''(t) \leqq 0$ or $y''(t) \geqq 0$.

In the first case, $y'(t) < 0$ is a nonincreasing function and for $c \geqq b$ there exists a positive constant K_1 such that $y'(t) < -K_1$ on $[c, \infty)$. By integrating this inequality from c to t we obtain

$$y(t) \leqq -K_1(t - c) + y(c) \to -\infty \qquad \text{for} \qquad t \to \infty$$

which is a contradiction for $y(t) > 0$ on $[a, \infty)$.

Now let $y''(t) \geqq 0$. Since $y''(t)$ is a strongly increasing function, there exists $d \geqq b$ and a positive constant K_2 such that $y''(t) > K_2$ on $[d, \infty)$. By integration from d to t,

$$y'(t) > K_2(t - d) + y'(d).$$

We see that $y'(t)$ has a zero on $[d, \infty)$, which is a contradiction.

Thus we have proved that $y(t)\,y'(t) > 0$ on $[a, \infty)$.

Lemma 3. *Let* $p(t) \geqq 0$, $q(t) < 0$, *and* $y(t)$ *be a nontrivial nonoscillatory solution of the equation* (L) *satisfying* $F(y(t)) > 0$ *on* $[a, \infty)$. *Then* $c \in [a, \infty)$ *exists such that* $y(t)\,y'(t) > 0$ *for all* $t \geqq c$.

Proof: Let $y(t)$ be any solution of (L) which is nonoscillatory. Let t_0 be its last zero. If $y(t)$ is nonvanishing on $[a, \infty)$, let t_0 be arbitrary. We can suppose without loss of generality that $y(t) > 0$ for all $t > t_0$.

We assert that the function $y'(t)$ has at most one zero on (t_0, ∞). Indeed, if $t_1 \in \in (t_0, \infty)$ is a zero of $y'(t)$, $F(y(t_1)) > 0$ and hence $y''(t_1) < 0$. Consequently t_1 is the unique zero.

Let $c > t_1 > t_0$. Then $y(t)\,y'(t) \neq 0$ holds on $[c, \infty)$ and the assertion follows from Lemma 2.

Lemma 4. *Let $p(t) \geqq 0$, $q(t) < 0$ and $p'(t) - 2q(t) \geqq 0$. If*

$$\int_a^\infty [p'(t) - 2q(t)]\,\mathrm{d}t = \infty$$

and $y(t)$ is a nontrivial solution of the equation (L) satisfying $F(y(t)) > 0$ on $[a, \infty)$, then $y(t)$ is an oscillatory solution.

Proof by contradiction: Let $y(t) \not\equiv 0$ be a nonoscillatory solution of the equation (L) and $F(y(t)) > 0$ on $[a, \infty)$. By Lemma 3 there exists $c \in [a, \infty)$ such that $y(t)\,y'(t) > 0$ on $[c, \infty)$. Without loss of generality we can suppose $y(t) > 0$. Then for arbitrary $d \geqq c$ there exists a positive constant K such that we can put $y(t) \geqq K$ on $[d, \infty)$. From Mammana's identity (M) it follows

$$F(y(t)) = F(y(d)) - \int_a^t [p'(s) - 2q(s)]\,y^2(s)\,\mathrm{d}s$$

$$\leqq F(y(d)) - K^2 \int_a^t [p'(s) - 2q(s)]\,\mathrm{d}s$$

and for $t \to \infty$ there is $F(y(t)) \to -\infty$, which is a contradiction with our supposition.

We have proved that $y(t)$ cannot be nonoscillatory under the given supposition.

Lemma 5. *Let $p(t) \geqq 0$, $q(t) < 0$ and $p'(t) - 2q(t) \geqq 0$. If*

$$\int_a^\infty [p'(t) - 2q(t)]\,\mathrm{d}t = \infty,$$

then the nontrivial solution $y(t)$ of the equation (L) is nonoscillatory iff $c \in [a, \infty)$ exists such that $F(y(c)) \leqq 0$.

Proof: Let $y(t)$ be a nontrivial solution of the equation (L). If $F(y(t)) > 0$ on $[a, \infty)$, then $y(t)$ is oscillatory by Lemma 4. Then $c \in [a, \infty)$ exists for nonoscillatory $y(t)$ such that $F(y(c)) \leqq 0$.

On the contrary, if $F(y(c_1)) \leqq 0$ for some $c_1 \in [a, \infty)$, then $F(y(t)) < 0$ on (c, ∞) since $F(y(t))$ cannot be a constant. Let us suppose that $y(t)$ has the root in $t_0 \in (c, \infty)$. Then $F((y(t_0)) = y'^2(t_0) \geqq 0$, which is a contradiction. The solution $y(t)$ must be nonoscillatory. Thus the assertion is proved.

Theorem 1. *Let $p(t) \geqq 0$, $q(t) < 0$ and $p'(t) - 2q(t) \geqq 0$. If*

$$\int_a^\infty [p'(t) - 2q(t)]\,\mathrm{d}t = \infty,$$

then the equation (L) has two linearly independent oscillatory solutions.

Proof: Let the solutions $u_1(t)$, $u_2(t)$, $u_3(t)$ of the equation (L) satisfy the initial conditions

$$u_i^{(j)}(a) = \delta_{i,j+1} = \begin{cases} 0, & i \neq j+1 \\ 1, & i = j+1 \end{cases} \qquad \begin{matrix} i = 1, 2, 3, \\ j = 0, 1, 2. \end{matrix}$$

Let $n > a$ be positive integers, b_{1n}, b_{3n} and c_{2n}, c_{3n} constants such that the solutions of equation (L) of the form

$$v_n(t) = b_{1n}u_1(t) + b_{3n}u_3(t),$$
$$w_n(t) = c_{2n}u_2(t) + c_{3n}u_3(t),$$
$$(b_{1n}^2 + b_{3n}^2 = c_{2n}^2 + c_{3n}^2 = 1)$$

satisfy $v_n(n) = w_n(n) = 0$. Then $F(v_n(n)) \geqq 0$, $F(w_n(n)) \geqq 0$ and since $F(y(t))$ cannot be a constant on intervals of the form $[t_0, \infty)$, there holds

(1) $\qquad F(v_n(t)) > 0,\ F(w_n(t)) > 0$ on $[a, b_n)$, where $b_n \to \infty$ as $n \to \infty$.

By Lemma 1 the sequence $\{n_k\}$ exists such that $v_{n_k}(t)$ converges for $n_k \to \infty$ on every finite subinterval from $[a, \infty)$ uniformly to the function $v(t)$ and there holds.

$$v^{(s)}(t) = b_1u_1^{(s)}(t) + b_3u_3^{(s)}(t), \qquad s = 0, 1, 2,$$
$$b_1^2 + b_3^3 = 1.$$

From (1) it follows that $F(v(t)) \geqq 0$ on $[a, \infty)$. As $F((yt))$ is a nonincreasing function and is not a constant on $[a, \infty)$, there must be $F(v(t)) > 0$ on $[a, \infty)$. In the contrary case $F(v(t))$ obtains negative values, which is a contradiction. We shall prove similarly that $F((w(t)) > 0$ and $c_2^2 + c_3^2 = 1$ on $[a, \infty)$.

Solutions $v(t)$, $w(t)$ are oscillatory by Lemma 4. Let the solutions $v(t)$, $w(t)$ be depend. As $b_1^2 + b_3^2 = c_2^2 + c_3^2 = 1$ is satisfied, there holds $v(t) = Ku_3(t)$ for some $K \neq 0$. Then however $v(t)$ is nonoscillatory by Lemma 5, because $F(u_3(a)) = 0$ by definition of $u_3(t)$, which is a contradiction.

We have proved that $v(t)$, $w(t)$ are linearly independent solutions; this completes the proof.

Theorem 2. *Let $p(t) \geqq 0$ be a bounded function, $q(t) < 0$,*

$$\int_a^\infty [p'(t) - q(t)]\, dt = \infty.$$

If $y(t)$ is a nontrivial nonoscillatory solution of the equation (L) satisfying $y'(t) \neq 0$ on $[a, \infty)$, then $y(t)$ is unbounded.

Proof: Let $y(t)$ be a nonoscillatory solution of the equation (L) satisfying $y'(t) \neq 0$ on $[a, \infty)$. Without loss of generality we can assume $y(t) > 0$ on $[a, \infty)$. By Lemma 2 there holds $y(t)\,y'(t) > 0$ on this interval. Then $c \in [a, \infty)$ and a positive constant K_1 exist such that we can put $y(t) \geqq K_1$ on $[c, \infty)$.

Let us suppose that $y(t)$ is a bounded solution. Since $p(t)$ is a bounded function by the supposition, positive constants K_2, K_3 exist such that $y(t) \leqq K_2$ and $p(t) \leqq K_3$ on $[c, \infty)$. By means of integration of the equation (L) within the limits c, t we obtain

$$y''(t) + p(t)\,y(t) - y''(c) - p(c)\,y(c) = \int_c^t [p'(s) - q(s)]\,y(s)\,\mathrm{d}s.$$

There holds

$$y''(t) + K_3K_2 + \text{const} \geqq \int_c^t [p'(s) - q(s)]\,y(s)\,\mathrm{d}s \geqq$$

$$\geqq K_1^2 \int_c^t [p'(s) - q(s)]\,\mathrm{d}s.$$

Hence we have $y''(t) \to \infty$ for $t \to \infty$. A positive constant N for $d \in [c, \infty)$ exists such that $y'(t) > N$ on $[d, \infty)$. By integration from d to t then $y(t) > N(t - \mathrm{d}) + y(d) \to \infty$ for $t \to \infty$, which is a contradiction. Then the solution $y(t)$ is unbounded.

So the assertion is proved.

Example: Let us consider the equation (L) on the interval $[2, \infty)$ for

$$p(t) = 1 - \frac{4}{3}t^{-2} > 0, \qquad q(t) = \frac{16}{27}t^{-3} - \frac{2}{3}t^{-1} < 0.$$

Further there holds

$$p'(t) - 2q(t) = \frac{4}{3}t^{-1} + \frac{40}{27}t^{-3} > 0$$

and

$$\int_2^\infty [p'(t) - 2q(t)]\,\mathrm{d}t = \infty.$$

By Theorem 1 this equation has two linearly independent oscillatory solutions

$$v(t) = t^{-1/3}\cos t, \qquad w(t) = t^{-1/3}\sin t$$

for which the functions F of Mammana's identity (M) are positive. Further linearly independent solution of this equation is nonoscillatory

$$u(t) = t^{2/3}, \qquad F(u(t)) \to -\infty \qquad \text{for } t \to \infty.$$

It can be easily verified that for $u(t)$ the suppositions of Theorem 2 are satisfied.

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