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MATRICES OF HOMOMORPHIC MAPPINGS

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Some methods for investigations of homomorphic mappings of direct products of algebras without zero-divisors are introduced in [3] and [4]. We can simplify this theory and give an effective implement for investigations of this homomorphisms – so called matrix representation. We can use it advantageously in applications (specially for extensions of mappings into homomorphisms). This theory is analogous in some aspects to the theory of matrix over rings in case when a binary operation $\oplus$ is associative. Special matrices of homomorphic mappings are investigated in [5], applications of them are contained in [6].

1.

In whole this paper a symbol $\mathfrak{A}$ denotes a class of algebras with the zero-element 0, binary operation $\oplus$ and a set Ω of n-ary operations fulfilling the following identities:

(i) $$0 \oplus a = a \oplus 0 = a,$$

(ii) $$\text{for each } \omega \in \Omega \text{ is } 00 \dots 0\omega = 0,$$

for arbitrary $A \in \mathfrak{A}$ and each $a \in A$.

An algebra $A \in \mathfrak{A}$ is called *without zero-divisors* if card $A > 1$ and if there exists $\Omega' \subseteq \Omega$, $\Omega' \neq \emptyset$ such that the arity of operations from Ω' is greater than 1 and for each $\omega \in \Omega'$:

(iii) $$a_1a_2 \dots a_n\omega = 0 \quad \text{iff} \quad a_i = 0 \quad \text{for at least one} \quad i \in \{1, \dots, n\}.$$

Operations from Ω' are called *regular*.

An algebra $A \in \mathfrak{A}$ is called *strongly pseudo-ordered* (see [3] and [4]) if there exists $\Omega'' \subseteq \Omega'$, $\Omega'' \neq \emptyset$ such that for each $\omega \in \Omega''$ the following holds:

(iv) $$a_1a_2 \dots a_n\omega = a_i, \qquad \text{where} \qquad i \in \{1, \dots, n\}.$$

Some characterizations of strongly pseudo-ordered algebras and some examples of them are given in [3] and [4]. Direct products of strongly pseudo-ordered algebras are for example atomic Boolean algebras, distributive lattices which are direct products of chains with the least or the greatest element and weakly associative lattices which are direct products of tournaments with the least or the greatest element (see [8] and [7]). Direct products of algebras without zero-divisors are certain l-groups (see [2]), Ω-groups and Ω-algebras without nilpotent elements (see [7]), any lattice ordered rings which are direct products of completely ordered rings, direct products of rings without zero-divisors and others (see [3], [4]).

Let $A_\tau \in \mathfrak{A}$ be algebras without zero-divisors for $\tau \in T = \{1, \dots, n\}$. Operations are denoted by the same symbols in all these algebras. Direct product of A_τ is denoted by $\prod_{\tau \in T} A_\tau$ or by $\prod_{\tau=1}^{n} A_\tau$. Let $A = \prod_{\tau \in T} A_\tau$, $A_\tau \in \mathfrak{A}$. By the symbol $\bar{A}_{\tau'}$ (or $\overline{\prod_{\tau \in T'} A_\tau}$ for $T' \subseteq T$) is denoted a subalgebra of A fulfilling $pr_{\tau'} \bar{A}_{\tau'} = A_{\tau'}$, $pr_\tau \bar{A}_{\tau'} = 0$ for $\tau \neq \tau'$ (or $pr_\tau \overline{\prod_{\tau \in T'} A_\tau} = A_\tau$ for $\tau \in T'$ and $pr_\tau \overline{\prod_{\tau \in T'} A_\tau} = 0$ for $\tau \in T - T'$, respectively). In [4] there is proved that the direct decomposition of an algebra $A \in \mathfrak{A}$ into algebras without zero-divisors is unique (if it exists) within to isomorphism.

Let A_τ, $B_\sigma \in \mathfrak{A}$ be algebras without zero-divisors for $\tau \in T$, $\sigma \in S$, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$, T and S being finite index sets and φ be a homomorphic mapping of A into B. We say that A, B, φ fulfil the assumption (P) if either 1° or 2° holds:

1° $\varphi(0_A) = 0_B$, where 0_A (or 0_B) is a zero of A (or B respectively), i.e. $pr_\tau 0_A = 0$, $pr_\sigma 0_B = 0$ for each $\tau \in T$, $\sigma \in S$.

2° A_τ, B_σ are strongly pseudo-ordered for all $\tau \in T$, $\sigma \in S$.

In [3] and [4] there is proved that the assumption (P) holds for all algebras mentioned above and for arbitrary homomorphism of them. In this paper we investigate homomorphisms of algebras which are finite direct products of algebras without zero-divisors and fulfil the assumption (P).

2.

Let A, $B \in \mathfrak{A}$. By the symbol $H(A, B)$ we denote the set of all homomorphic mappings of A into B, o is the *zero-homomorphism* of A into B, i.e. $o(A) = 0 \in B$. Evidently $H(A, B) \neq \emptyset$ for arbitrary A, $B \in \mathfrak{A}$, because $o \in H(A, B)$. Let $a_i \in A$ for $i = 1, \dots, m$. By $\circ\sum_{i=1}^{m} a_i$ we denote the element $(\dots((a_1 \oplus a_2) \oplus a_3) \oplus \dots) \oplus a_m$. If $\oplus$ is associative, then $\circ\sum_{i=1}^{m} a_i = a_1 \oplus a_2 \oplus \dots \oplus a_m$. If $a_1 = \dots = a_{j-1} = a_{j+1} = \dots = a_m = 0$, then $\circ\sum_{i=1}^{m} a_i = a_j$.

Definition 1. A homomorphic mapping $\varkappa$ of an algebra A into B is said to be *idempotent with the map* h if $\varkappa(A) = h \in B$.

If $\varkappa$ is an idempotent mapping of A into B with the map h, then h is an idempotent element of B (because $\varkappa$ is a homomorphism). Evidently, each zero-homomorphism is an idempotent mapping with the map 0.

Definition 2. Let $A_i, B_j \in \mathfrak{A}$, $(i = 1, \ldots, m,\ j = 1, \ldots, n)$, $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$. Let F be a matrix of the type m/n which has in the i-th row and the j-th column an element $f_{ij} \in H(A_i, B_j)$. Let f be a mapping of A into B defined by the rule:

$$pr_j f(a) = \circ \sum_{i=1}^{m} f_{ij}(pr_i a) \qquad \text{for each } a \in A.$$

We say that f is *represented by the matrix* F.

The following lemma is evident:

Lemma. *Let f be a mapping of $A = \prod_{i=1}^{m} A_i$ into $B = \prod_{j=1}^{n} B_j$ represented by the matrix F and $A_i, B_j \in \mathfrak{A}$ be without zero-divisors. Then f is an idempotent mapping of A into B with the map h if and only if all elements in the j-th column are idempotent mappings with the map $pr_j h$ for each $j \in \{1, \ldots, n\}$.*

Especially $f = o$ iff $f_{ij} = o$ for all i, j.

Theorem 1. *Let $A_i, B_j \in \mathfrak{A}$ be algebras without zero-divisors and $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$, φ be a mapping of A into B represented by a matrix F (of the type m/n) and the assumption (P) be true. Then φ is a homomorphism if and only if all elements in the j-th column except at most one are equal to idempotent mappings with the map $pr_j\varphi(0_A)$ for each $j = 1, \ldots, n$.*

Proof. If all elements in the j-th column are idempotent mappings with the map $pr_j\varphi(0_A)$, then $pr_j\varphi(A) = pr_j\varphi(0_A)$, thus $pr_j \cdot \varphi$ is a homomorphism. If f_{kj} is the only element in the j-th column different from idempotent mapping with the map $pr_j\varphi(0_A)$, then $pr_j \cdot \varphi(a) = \circ\sum_{i=}^{m} f_{ij}(pr_i a) = f_{kj}(pr_k a)$ for each $a \in A$, as follows directly from the definition 2, because $pr_j\varphi(0_A)$ is an idempotent element and $\varphi(0_A)$ is a zero-element of $\varphi(A)$. Thus $pr_j \cdot \varphi$ is a homomorphism again, i.e. φ is a homomorphic mapping of A into B.

Conversely, let φ be a homomorphic mapping of A into B represented with the matrix F and let F have at least two elements f_{ik}, f_{jk}, $i \neq j$ in the k-th column, which are not idempotent mappings with the map $pr_k\varphi(0_A)$. Then there exist elements $\bar{a}_i \in \bar{A}_i$, $\bar{a}_j \in \bar{A}_j$ such that

$$f_{ik}(pr_i\bar{a}_i) \neq pr_k\varphi(0_A) \neq f_{jk}(pr_j\bar{a}_j).$$

Let ω be a direct product of suitable regular operations, then

$$\varphi(0_A) = \varphi(\bar{a}_i\bar{a}_j \ldots \bar{a}_j\omega) = \varphi(\bar{a}_i)\,\varphi(\bar{a}_j) \ldots \varphi(\bar{a}_j)\,\omega.$$

By (P) either $\varphi(0_A) = 0_B$ or A_i, B_j are strongly pseudo-ordered. In the first case $\varphi(\bar{a}_i) \neq 0_B \neq \varphi(\bar{a}_j)$ and

$$\varphi(\bar{a}_i)\, \varphi(\bar{a}_j) \dots \varphi(\bar{a}_j)\, \omega \neq 0_B = \varphi(0_A),$$

which is a contradiction. In the second case we have for a direct product ω of suitable operation from Ω'': $pr_k[\varphi(\bar{a}_i)\, \varphi(\bar{a}_j) \dots \varphi(\bar{a}_j)\, \omega] = pr_k\varphi(\bar{a}_i)\, pr_k\varphi(\bar{a}_j) \dots pr_k\varphi(\bar{a}_j)\, \omega = pr_k\varphi(\bar{a}_i)$ or $pr_k\varphi(\bar{a}_j)$. This element is different from $pr_k\varphi(0_A)$, i.e. $\varphi(\bar{a}_i)\, \varphi(\bar{a}_j) \dots \varphi(\bar{a}_j)\, \omega \neq \varphi(0_A)$ which is a contradiction again.

q.e.d.

Remark. For surjective homomorphic mappings of A onto B the assumption (P) is fulfilled automatically, because $\varphi(0_A) = 0_B$, which is easy to prove (see Lemma in [3]).

Corollary 2. Let A_i, $B_j \in \mathfrak{A}$ *be algebras without zero-divisors,* $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$ *and* φ *be a mapping of* A *into* B *fulfilling* $\varphi(0_A) = 0_B$ *represented by the matrix* F. *Then* φ *is a homomorphism if and only if* F *has in each column at least one non-zero homomorphism.*

From it follows the theorem 1 in [5].

Theorem 3. *Let* φ *be a homomorphic mapping of* $A = \prod_{i=1}^{m} A_i$ *into* $B = \prod_{j=1}^{n} B_j$, A_i, $B_j \in \mathfrak{A}$ *be algebras without zero-divisors and let the assumption* (P) *be true. Then there exists just one matrix* F *(of the type* m/n*) representing* φ.

Proof. Let N' be a subset of $N = \{1, \dots, n\}$ such that $pr_k\varphi(A) = pr_k\varphi(0_A)$ for $k \in N - N'$ and $pr_k\varphi(A) \neq pr_k\varphi(0_A)$ for $k \in N'$. Let us construct a matrix F of type m/n by the following way:

(a) If $j \in N - N'$, then f_{ij} is equal to an idempotent mapping of A_i into B_j with the map $pr_j\varphi(0_A)$ for $i = 1, \dots, m$.

(b) If $j \in N'$, then there exists just one i such that $pr_j\varphi(A) = pr_j\varphi(\bar{A}_i)$ (by the theorem 7 in [4]). Put $f_{ij} = pr_j \cdot \varphi\,|_{Ai} \cdot p$, where p is a natural isomorphism of A_i onto $\bar{A}_i$ (see [3]), and all other elements of the j-th column are idempotent mappings with the map $pr_j\varphi(0_A)$. From the definition 2 and the theorem 1 follows directly the assertion of the theorem 3, because the unicity of F is evident.

3.

We can introduce operations for matrices of homomorphisms analogous to matrix multiplication and multiplication of matrix by vector for matrices over ring. Let A_i, B_j, $C_k \in \mathfrak{A}$ be algebras without zero-divisors and $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$, $C = \prod_{k=1}^{p} C_k$. If $a \in A$, we can write a in the form $a = (a_1, \dots, a_m)$, so called "*vector form*". Then $pr_i a = a_i$. Analogously we can write $b = (b_1, \dots, b_n)$, $c = (c_1, \dots, c_p)$.

Definition 3. Let a matrix F of type m/n represent homomorphism f of A into B, a matrix R of type n/p represent homomorphism g of B into C. By the *product of element $a \in A$ with the matrix F* we mean an element $b \in B$ such that $b_j = \circ \sum_{i=1}^{m} f_{ij}(a_j)$, symbolically $a \,.\, F = b$. By the *product of matrices F, G* we mean a matrix H of the type m/p which has elements $h_{sq} \in H(A_s, C_q)$ such that

$$a \in A \Rightarrow h_{sq}(a_s) = \circ \sum_{j=1}^{n} g_{jq}[f_{sj}(a_s)]. \quad \text{Symbolically} \quad H = F \,.\, G.$$

It is clear that this operations are analogous to the matrix multiplication of matrices over ring, where $\oplus$ is an analogon to the ring addition $+$ and mappings product (i.e. mappings superposition) is an analogon to the ring multiplication. If $\oplus$ is associative, we can use the rule of multiplication "rows of matrix F by columns of matrix G" for multiplication of matrices of homomorphisms.

Theorem 4. *Let A_i, B_j, $C_k \in \mathfrak{A}$ be algebras without zero-divisors, $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$, $C = \prod_{k=1}^{p} C_k$, $a \in A$. Let φ be a homomorphic mapping of A into B represented by the matrix F, ψ be a homomorphic mapping of B into C represented by the matrix G and let the assumption (P) be true for A, B, φ and B, C, ψ. Let $H = F \,.\, G$ and $b = a \,.\, F$. Then $b = \varphi(a)$ and H represents the mapping $\xi = \psi \,.\, \varphi$ of A into C.*

Proof. We have $b = \varphi(a)$ directly from definitions 2 and 3. Let $g_{j_k k}$ is either only one mapping in the k-th column, which is not idempotent mapping with the map $pr_k\psi(0_B)$ or $g_{j_k k} = pr_k\psi$ if there does not exist mapping with this property. Analogously for $f_{i_j j}$. Then

$$\begin{aligned} \xi(a) &= (\psi \,.\, \varphi)\,(a) = \psi[\varphi(a_1, \dots, a_m)] = \psi(a \,.\, F) = \\ &= \psi[f_{i_1 1}(a_{i_1}), \dots, f_{i_n n}(a_{i_n})] = (a \,.\, F) \,.\, G = \\ &= [g_{j_1 1}(f_{i_{j_1} j_1}(a_{i_{j_1}})], \dots, g_{j_p p}[f_{i_{j_p} j_p}(a_{i_{j_p}})] = a \,.\, (F \,.\, G) = a \,.\, H, \end{aligned}$$

because F and R have in each column at least one mapping which is not idempotent with the map equal to projection of $\varphi(0_A)$ or $\varphi(0_B)$ respectively.

From the theorems 3 and 4 it follows immediately:

Corollary 5. *Matrix multiplication of the matrices of homomorphisms is associative.*

4.

It follows directly from the theorem 4 and corollary 5 that matrices of type n/n (i.e. *square matrices*) representing homomorphic mappings of $A = \prod_{i=1}^{n} A_i$ into itself form a semigroup with respect to matrix multiplication isomorphic to the semigroup of endomorphisms of the algebra A fulfilling (P); A_i are without zero-divisors. Thus the order of this matrix semigroup is equal to the number of endomorphisms of algebra A fulfilling (P).

Let the assumption (P) be true. Let us denote $H^*(A_i, B_j) = H(A_i, B_j)$ if 2° of (P) is true and $H^*(A_i, B_j)$ is the set of all homomorphisms from $H(A_i, B_j)$ satisfying $0 \to 0$ if 1° of (P) is true. Then we can prove the following:

Theorem 6. *Let $A_i, B_j \in \mathfrak{A}$ be algebras without zero-divisors, $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$, card $H^*(A_i, B_j) = p_{ij} < \aleph_0$ and c_j be the cardinality of the set of idempotent elements of the algebra B_j. Put $k_j = \bar{c}_j \, . \max (1, \sum_{i=1}^{m} (p_{ij} - \bar{c}_j))$, where $\bar{c}_j = 1$ if 1° of (P) for all $\varphi \in H(A; B)$ is true and $\bar{c}_j = c_j$ in other case. Then there exist just $s = k_1 k_2 \ldots k_n$ homomorphisms of A into B satisfying (P).*

Proof. By theorem 3 the number of all homomorphisms of A into B fulfilling (P) equals to the number of all matrices of mappings having all elements in each column with except at most one equal to the idempotent mappings with the map equal to projection of $\varphi(0_A)$. Denote $h_j = pr_j \varphi(0_A)$ for $\varphi \in H(A, B)$. From assumptions of theorem we obtain that there exist just c_j idempotent mappings of A_i into B_j. If 2° of (P) is true, then there exist $p_{ij} - c_j$ homomorphisms of A_i into B_j which are not idempotent with the map $pr_j \varphi(0_A)$, if only 1° of (P) is true, then there exist just $p_{ij} - 1$ non-zero homomorphisms of A_i into B_j fulfilling $0 \to 0$. If $p_{ij} = \bar{c}_j$, then there can be placed only idempotent mappings with the map h_j in the j-th column, thus $k_j = \bar{c}_j = c_j = \bar{c}_j \, . \max (1, \sum_{i=1}^{m} (p_{ij} - \bar{c}_j))$. If $p_{ij} > \bar{c}_j$, then there exist just $\sum_{i=1}^{m} (p_{ij} - \bar{c}_j)$ possibilities of placing of mappings which are not idempotent with the map h_j in the j-th column, other elements are idempotent mappings with the same map h_j, i.e. $\bar{c}_j$ possibilities, thus $k_j = \bar{c}_j \, . \sum_{i=1}^{m} (p_{ij} - \bar{c}_j) = \bar{c}_j \, . \max (1, \sum_{i=1}^{m} (p_{ij} - \bar{c}_j))$. It is evident that $s = k_1 \, . \, k_2 \ldots k_n$. q.e.d.

Corollary 7. *Let $A_i \in \mathfrak{A}$ be without zero-divisors, $A = \prod_{i=1}^{n} A_i$, p_{ij} be the number of all homomorphisms of A_i into A_j satisfying $0 \to 0$, $p_{ij} < \aleph_0$. Then there exist just $s = k_1 k_2 \ldots k_n$ endomorphisms of A satisfying $0_A \to 0_A$, where $k_j = \max(1, \sum_{i=1}^{n} (p_{ij} - 1))$. Let $A_i \in \mathfrak{A}$ be strongly pseudo-ordered algebras, c_j be a number of idempotent elements of A_j and p_{ij} = card $H(A_i, B_j)$, $p_{ij} < \aleph_0$. Then there exist just $s = k_1 k_2 \ldots k_n$ endomorphisms of A, where $k_j = c_j \, . \max (1, \sum_{i=1}^{n} (p_{ij} - c_j))$.*

For each isomorphism φ of A onto B we have $\varphi(0_A) = 0_B$, thus:

Corollary 8. *Let $A_i, B_j \in \mathfrak{A}$ be algebras without zero-divisors, $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$*

and $q_{ij} < \aleph_0$ *be the number of all isomorphisms of* A_i *onto* B_j. *Then there exist just* $s = k_1 k_2 \ldots k_n$ *isomorphisms of* A *onto* B, *where* $k_j = \sum_{i=1}^{m} q_{ij}$.

From the theorem 5 and corollary 7 and 8 there follows the theorem 3 in [5] and its corollary.

5.

Now we can state assertions on relations between matrix representation of homomorphisms and direct decompositions of these homomorphisms. Theorems on direct decompositions of algebras which are direct products of algebras without zero-divisors are formulated in [4]. A consequence of theorems 7 and 8 in [4] and the theorem 2 in this paper is the following one:

Theorem 9. *Let* $A_i, B_j \in \mathfrak{A}$ *be algebras without zero-divisors,* $A = \prod_{i=1}^{m} A_i$, $B = \prod_{j=1}^{n} B_j$, φ *be a homomorphic mapping of* A *into* B *and let the assumption* (P) *be true. Then there exists* $M' \neq \emptyset$, $M' \subseteq \{1, \ldots, m\}$ *such that* $\varphi(A) = \varphi(A^*)$, *where* $A^* = \prod_{i \in M'} A_i$, *and* $\varphi|_{A^*} = \prod_{j=1}^{n} \varphi_j$ *if and only if the matrix* F *representing* φ *has in the i-th row at most one mapping* φ_j *which is not idempotent with the map equal to the projection of* $\varphi(0_A)$ *for* $i = 1, \ldots, m$.

The direct product $\prod_{j=1}^{n} \varphi_j$ of mappings φ_j is introduced in [3]. Following statement is a consequence of the theorem 9.

Corollary 10. *Let* $A_i, B_i \in \mathfrak{A}$ *be algebras without zero-divisors,* $A = \prod_{i=1}^{m} A_i$, $B = \prod_{i=1}^{m} B_i$, φ *be a homomorphic mapping of* A *into* B *and let the assumption* (P) *be true. Then* $\varphi = \prod_{i=1}^{m} \varphi_i$, *where* $\varphi_i \in H(A_i, B_{\pi(i)})$ *and* π *is a permutation of the set* $\{1, \ldots, m\}$ *if and only if the matrix* F *representing* φ *has in each row and in each column just one mapping which is not idempotent with the map equal to projection of* $\varphi(0_A)$.

If $A_i, B_i \in \mathfrak{A}$ are strongly pseudo-ordered and φ is an arbitrary homomorphism of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{i=1}^{n} B_i$, then the assumption (P) is satisfied. If φ is a surjective homomorphism, then $\varphi = \prod_{i=1}^{n} \varphi_i$, where $\varphi_i \in H(A_i, B_{\pi(i)})$ by the theorem 6 in [3]. From it and the corollary 10 we obtain:

Corollary 11. *Let* $A_i, B_i \in \mathfrak{A}$ *be strongly pseudo-ordered algebras and* $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$. *The matrix* F *representing a surjective homomorphic mapping* φ *of* A *onto* B *has in each row and each column just one element which is not idempotent mapping with the map equal to the projection of* $\varphi(0_A)$.

6.

Whole this theory of matrix representation of homomorphic mappings can be generalized for infinite direct products of algebras without zero-divisors, i.e. for so called N-algebras (see [3] and [4]). We can introduce a concept of infinite matrix and state analogons of theorems 1, 2, 3, 5, 6, 7, 8 and 10. For validity of the analogon of theorem 4 we can define an infinite "*sum*" (with respect to $\oplus$). This is possible, because almost all elements in an infinite "*sum*" are equal to idempotent mappings with the map h_j and $h_j \oplus h_j = h_j$ (see [3]). An infinite analogon of the theorem 9 is the theorem 8 in [4].

The simple applications of matrix representation of homomorphisms is shown in [5]. For example, for some algebras from $\mathfrak{A}$ the sets $H(A_i, B_j)$ are very simple. Each atomic Boolean algebra (see [1]) is a direct product of two-elemented Boolean algebras $\{0,1\}_B$. For $A_i = B_j = \{0,1\}_B$ we have $H(A_i, B_j) = \{o, id, z\}$, where o is a zero-homomorphism, id is the identical isomorphism and z is an idempotent homomorphism with the map 1. Analogously, for simple rings, fields, completely ordered groups, chains with a bound element the sets $H(A_i, B_j)$ are very simple and we can easy (by the theorems 4.6 and 9 and their consequences) investigate homomorphic mappings of direct products of these algebras. For all algebras mentioned above namely (P) holds automatically.

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