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ON THE IMPOSSIBILITY TO CONSTRUCT DIAMETRICALLY CRITICAL GRAPHS BY EXTENSIONS

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The graphs considered in this paper are undirected, finite, without loops and multiply edges.

A graph $G$ is called $e$-critical ($v$-critical) if its diameter is changed after removing any edge (any vertex and all edges incidental with it), respectively. These graphs were studied in [4], [8], [3]. An $\omega_d$-graph is a graph of diameter $d \geq 2$ and girth at least $d + 2$. An $\bar{\omega}_d$-graph is a graph of diameter $d \geq 2$, girth at least $d + 3$ and minimum degree at least two. Obviously, every $\omega_d$-graph is $e$-critical, (see [4]) and every $\bar{\omega}_d$-graph is $v$-critical, (see [3]).

Let $G$ be a graph. Then $V(G)$ will denote the vertex set of $G$, $d(G)$ the diameter of $G$, $d_G(u, v)$ the distance between the vertices $u, v$ of $G$, $N(z)$ the neighbourhood of a vertex $z$ of $G$ and $| A |$ the cardinality of a set $A$. Definitions of notions not included here can be found in [6].

**Definition 1.** Let $G$ and $Q$ be vertex disjoint graphs and let $R$ be a graph such that $V(R) = V(G) \cup V(Q)$ and $G$ and $Q$ are induced subgraphs of $R$. Then we say that the graph $R$ is a $Q$-extension of $G$, or that the graph $R$ is a connection of graphs $G$ and $Q$ and the related notation is $R = G \oplus Q$.

In the sequel let $\mathfrak{A}$ be a certain class of graphs. Let $\mathfrak{P}$ and $\mathfrak{S}$ be finite sets of graphs not necessarily disjoint.

**Definition 2.** We say that a graph $G$ can be constructed from a set $\mathfrak{P}$ by extensions $\mathfrak{S}$ if either $G \in \mathfrak{S}$ or there exists a finite sequence of graphs $H_0, H_1, \ldots, H_{n-1}, H_n$ such that $H_0 \in \mathfrak{P}$, $H_n = R$, $H_i \in \mathfrak{A}$ and $H_{i+1}$ is a $Q$-extension of $H_i$ for some graph $Q \in \mathfrak{S}$, where $1 \leq i \leq n - 1$.

Analogously a class of graphs $\mathfrak{A}$ can be constructed from a set $\mathfrak{P}$ by extensions $\mathfrak{S}$ if every graph $R \in \mathfrak{A}$ can be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$.

In [5] it is proved that $\omega_2$-graphs and $e$-critical graphs of diameter $d \geq 2$ cannot be constructed from a set $\mathfrak{P}$ by extensions $\mathfrak{S}$. We shall prove the same results for $\omega_d$-graphs and $\bar{\omega}_d$-graphs, where $d \geq 3$ and for $v$-critical graphs of diameter $d \geq 2$. We note that many classes of graphs are constructed from a finite set of primitive graphs by using finite set of some operations, e.g. [6], [7], [9].

Now we describe two constructions of graphs that will be used later.
**Example 1.** Let \( k \geq 4, m \geq 4 \) be given integers. The following \( \omega_2 \)-graph \( D = D(k, m) \) is constructed in [5]. The vertex set of \( D \),

\[
V(D) = \{v_0, v_1, \ldots, v_k\} \cup \{u_1, u_2, \ldots, u_{k-2}\} \cup \bigcup_{i=1}^{k} X_i, \quad \text{where } X_i = \{x_{i1}, x_{i2}, \ldots, x_{im}\}.
\]

Let us put \( X_{ij} = X_i - \{x_{ij}\} \), where \( i = 1, 2, \ldots, k; \ j = 1, 2, \ldots, m \). The edge set of \( D \) is given by setting the neighbourhood to every vertex of \( D \).

\[
\begin{align*}
N(v_0) &= \{v_1, v_2, \ldots, v_k\}; \\
N(v_i) &= \{v_0, u_1, u_2, \ldots, u_{k-2}\} \cup X_i, \quad \text{for } i = 1, 2; \\
N(u_j) &= \{v_0, u_{j-2}\} \cup X_j, \quad \text{for } j = 3, 4, \ldots, k; \\
N(u_i) &= \{v_1, v_2, v_{i+2}\} \cup \bigcup_{p=3, p \neq i+2}^{k} X_p, \quad \text{for } i = 1, 2, \ldots, k-2; \\
N(x_{1i}) &= \{v_1\} \cup \{x_{2i}, x_{3i}, \ldots, x_{ki}\}, \quad \text{for } i = 1, 2, \ldots, m; \\
N(x_{2i}) &= \{v_2, x_{1i}\} \cup \bigcup_{j=3}^{k-2} X_{ji}, \quad \text{for } i = 1, 2, \ldots, m; \\
N(x_{ji}) &= \{v_j, x_{1i}\} \cup X_{2i} \cup \bigcup_{p=1}^{k} \{u_p\}, \quad \text{for } j = 3, 4, \ldots, k; i = 1, \ldots, m;
\end{align*}
\]

The sketch of this graph is in Fig. 1. It is clear that \( D \) is \( e \)-critical graph. The graph \( D \) is also \( v \)-critical one because after deleting the vertex:

- \( v_0 \) it would be \( d(v_0, v_i) > 2 \), for \( i = 4, 5, \ldots, k \);
- \( v_i \) it would be \( d(v_0, x_{ij}) > 2 \), for \( i = 1, \ldots, k; j = 1, 2, \ldots, m \);
- \( w_i \) it would be \( d(v_{i+2}, x_{rs}) > 2 \), for \( i = 1, 2, \ldots, k-2; \ r = 3, 4, \ldots, k; \ r \neq i+2; \ s = 1, 2, \ldots, m \);
- \( x_{1i} \) it would be \( d(x_{2i}, x_{ri}) > 2 \), for \( i = 1, 2, \ldots, m; \ r = 3, \ldots, k \);
- \( x_{2i} \) it would be \( d(x_{1i}, z) > 2 \), for every \( z \in \bigcup_{j=3}^{k} X_{ji}, \ i = 1, 2, \ldots, m \);
- \( x_{ri} \) it would be \( d(v_r, x_{1i}) > 2 \), for \( r = 3, 4, \ldots, k; \ i = 1, 2, \ldots, m \);

**Example 2.** Let \( k \geq 3 \) be a prime number. We shall construct a regular graph \( C = C(k) \) of degree \( k + 1 \), diameter three and girth six. Thus the graph \( C \) will be \( e \)-critical and \( v \)-critical one. The vertex set of \( C \),

\[
V(C) = U \cup V \cup \bigcup_{i=1}^{k} X_i \cup \bigcup_{i=1}^{k} Z_i, \quad \text{where } U = \{u_0, u_1, \ldots, u_k\},
\]

\[
V = \{v_0, v_1, \ldots, v_k\}; \ X_i = \{x_{i1}, x_{i2}, \ldots, x_{im}\}; \ Z_i = \{z_{i1}^1, z_{i2}^1, \ldots, z_{im}^1\}.
\]

It yields \( |V(C)| = 2(k^2 + k + 1) \). The edge set of \( C \) is determined by setting the neighbourhood to every vertex:

\[
N(v_0) = \{v_1, v_2, \ldots, v_k\}; \\
N(v_i) = \{v_0, u_1, u_2, \ldots, u_{k-2}\} \cup X_i, \quad \text{for } i = 1, 2; \\
N(u_j) = \{v_0, u_{j-2}\} \cup X_j, \quad \text{for } j = 3, 4, \ldots, k; \\
N(u_i) = \{v_1, v_2, v_{i+2}\} \cup \bigcup_{p=3, p \neq i+2}^{k} X_p, \quad \text{for } i = 1, 2, \ldots, k-2; \\
N(x_{1i}) = \{v_1\} \cup \{x_{2i}, x_{3i}, \ldots, x_{ki}\}, \quad \text{for } i = 1, 2, \ldots, m; \\
N(x_{2i}) = \{v_2, x_{1i}\} \cup \bigcup_{j=3}^{k-2} X_{ji}, \quad \text{for } i = 1, 2, \ldots, m; \\
N(x_{ji}) = \{v_j, x_{1i}\} \cup X_{2i} \cup \bigcup_{p=1}^{k} \{u_p\}, \quad \text{for } j = 3, 4, \ldots, k; i = 1, \ldots, m;
\]
\[ N(u_0) = \{u_1, u_2, \ldots, u_k\} \cup \{v_0\} ; \]
\[ N(v_0) = \{v_1, v_2, \ldots, v_k\} \cup \{u_0\} ; \]
\[ N(u_i) = X_i \text{ and } N(v_i) = Z_i, \text{ for } i = 1, 2, \ldots, k ; \]

In addition, let \( r = 1, 2, \ldots, k \); \( i = 1, 2, \ldots, k \) and let the arithmetical operations in indices be computed modulo \( k \). Then we put

\[ N(z_1^r) = \{v_1\} \cup \{x_r^1, x_r^2, \ldots, x_r^k\} , \]
\[ N(z_2^r) = \{v_2\} \cup \{x_r^1, x_r^2, \ldots, x_r^{k+1}\}. \]
Analogically in general form

\[ N(z_i) = \{v_i\} \cup \{x_{r+i}^1, x_{r+2(i-1)}, \ldots, x_{r+(k-1)(i-1)}\}, \]

\[ N(u_i) = \{u_i\} \cup \{z_{r-i}^1, z_{r-2(i-1)}, \ldots, z_{r-(k-1)(i-1)}\}. \]

This construction is illustrated by the graph \( C = C(3) \) in Fig. 2. If \( k \) is not a prime number, then the graph \( C(k) \) contains a 4-angle. From the construction it follows that \( C(k) \) is a regular graph of degree \( k + 1 \). One can verify that the diameter of \( C(k) \) is equal to three and its girth is six by verifying separate cases.

Fig. 2
Example 3. Let $d \geq 4$, $m \geq 2$, $k \geq 2$ be given integers. Let us put $l = \left(\frac{d}{2}\right)$, $l^* = d - \left\lfloor \frac{d}{2} \right\rfloor$, where $\lfloor x \rfloor$ is the integer part of $x$. We construct an $\omega_d$-graph $B = B(k, m)$ as follows: $V(B) = \{w\} \cup \bigcup_{i=1}^{l^*} X_i \cup \bigcup_{i=1}^{l-1} Y_i \cup Z$, where $Y_i = \{x_i\}_{s=1}^k$, $Z = \{z_i\}_{i=1}^m$, $Y_r = \bigcup_{s=1}^{m} Y_r^{r,s}$, for $Y_r^{r,s} = \{y_r^{r,s}\}_{j=1}^m$, $r = 1, 2, \ldots, l - 1$. The edge set of $B$ is given again by the neighbourhoods. If $l \geq 3$, then we put: $N(w) = X_1$;

$$N(x_j) = \{w, x_j\}, \quad \text{for} \quad j = 1, 2, \ldots, k;$$
$$N(x_j) = \{x_{j-1}, x_{j+1}\}, \quad \text{for} \quad i = 2, 3, \ldots, l^* - 1; j = 1, 2, \ldots, k;$$
$$N(x_i^{l^*}) = \{x_{i}^{l-1,}, X_i^{l,1}\}, \quad \text{for} \quad i = 1, 2, \ldots, k;$$
$$N(y_j^{i+1}) = \{x_{i}, y_{j+1}\}, \quad \text{for} \quad i = 1, 2, \ldots, k; j = 1, \ldots, m;$$
$$N(y_i^{r,s}) = \{y_{i-1}, y_{i+1}^{r,s}\}, \quad \text{for} \quad r = 2, \ldots, l - 2; s = 1, \ldots, k; i = 1, \ldots, m;$$
$$N(y_i^{l-2,s}) = \{y_i^{l-2,s}, z_i\}, \quad \text{for} \quad i = 1, 2, \ldots, m; s = 1, 2, \ldots, k;$$

Fig. 3
If \( l = 2 \), then we replace the last three formulas by the following one:
\[
N(y_i^l, s) = \{x^*_i, z_i\}, \quad \text{for} \quad i = 1, \ldots, m; \\
N(z_i) = \{y_i^l, s\}, \quad \text{for} \quad i = 1, 2, \ldots, m.
\]
This construction is illustrated in Fig. 3.

One can verify that the diameter of \( B \) is equal to \( d \) and the girth of \( B \) is equal to \( 2d \) for \( d \) even and \( 2d - 2 \) for \( d \) odd. The graph \( B \) is an \( \omega_d \)-graph because its minimum degree is two and its girth is at least \( d + 3 \). Thus \( B \) is also an \( \omega_d \)-graph, e-critical and v-critical graph.

Now we prove the following theorem.

**Theorem 1.** Let \( \mathcal{P}, \mathcal{S} \) be finite sets of graphs. Let \( d \) be an integer. Then the class of e-critical graphs of diameter \( d \geq 2 \), v-critical graphs of diameter \( d \geq 2 \), \( \omega_d \)-graphs for \( d \geq 2 \) and \( \omega_d \)-graphs, for \( d \geq 3 \) cannot be constructed from the set \( \mathcal{P} \) by extensions \( \mathcal{S} \).

**Proof.** Let \( N \) be an integer greater than the number of vertices of any graph of the sets \( \mathcal{P} \) and \( \mathcal{S} \).

1. Let \( d = 2 \). The assertion of Theorem 1 can be proved quite analogously as in [5] by using v-critical graphs \( D(k, m) \) for \( k, m > N + 2 \). We do not repeat it for the briefness.

2. Let \( d = 3 \) and let \( k \) be a prime number greater than \( N \). We prove that \( \omega_3 \)-graph \( C(k) \) cannot be constructed from \( \mathcal{P} \) by extensions \( \mathcal{S} \). Then the class of \( \omega_3 \)-graphs cannot be constructed from any set \( \mathcal{P} \) by some extensions \( \mathcal{S} \), since there exist an infinite number of prime numbers \( k > N \).

If a graph \( C = C(k) \) can be constructed from \( \mathcal{P} \) by extensions \( \mathcal{S} \), then \( C = G \oplus Q \), where either \( G \in \mathcal{P}, \ Q \in \mathcal{S} \) or \( Q \in \mathcal{S} \) and \( G \) is an \( \omega_3 \)-graph. The case \( G \in \mathcal{P}, \ Q \in \mathcal{S} \) never occurs since \( |V(C)| > 2N \). Therefore the second case occurs. If a vertex \( u \) of \( C(k) \) belong to \( Q \), then at most one vertex of the set \( N(u) \) belongs to \( G \), since \( d(x, y) > 3 \) for every \( x, y \in N(u) \) in the graph \( G - u \). (This fact follows from the construction of \( C \).) So the graph \( Q \) contains at least \( 1 + (|N(u)| - 1) \) vertices, which is impossible as \( |N(u)| = k + 1 > N > |V(Q)| \). Thus \( \omega_3 \)-graphs cannot be constructed from \( \mathcal{P} \) by extensions \( \mathcal{S} \). The mentioned proof also proves the same assertion for \( \omega_3 \)-graphs, e-critical and v-critical graphs of diameter \( 3 \), since we used only the properties of \( C(k) \) diameter.

3. Let \( d \geq 4 \) and let \( k, m > N + 2 \). The \( \omega_d \)-graph \( B(k, m) \) is not a connection \( G \oplus Q \), where \( G \in \mathcal{P}, \ Q \in \mathcal{S} \) because \( |V(B)| > 2N \). Let \( B = G \oplus Q \), where \( G \) is an \( \omega_d \)-graph and \( Q \in \mathcal{S} \).

If \( w \in V(Q) \), then at least \( |X_1| - 1 = k - 1 \) vertices would belong to \( V(Q) \) since \( d(x, y) > d \) for every \( x, y \in N(w) = X_1 \) in the graph \( B - w \). This contradicts the fact \( |V(Q)| < N < k - 2 \). Therefore \( w \in V(G) \) is valid.

The vertex \( x^*_i \in V(G) \), where \( 1 \leq i \leq l^*, \ 1 \leq s \leq k \), because in the reverse case there would be \( d_0(w, z) > d \) for every \( z \in Y^{l-1,s} \) and so the set \( Y^{l-1,s} \) belongs to \( V(Q) \) which is in contradiction with \( |V(Q)| < N \).
The vertex \( y_j^s \in V(G) \), where \( 1 \leq r \leq l - 1, 1 \leq s \leq k, 1 \leq j \leq m \) since in the reverse case there would be \( d_{=}(x^2, y_j^{(r,s)}) > d \) for \( i \neq s, i = 1, 2, \ldots, k \) and hence the set \( \{y_j^{(r,s)}\}_{r=s+1}^{l-1} \) belongs to \( V(Q) \), which is impossible.

Finally, we have \( z_i \in V(G) \) for \( 1 \leq i \leq m \) since in the reverse case there would be \( d_{=}(y_l^{(r,s)}, y_r^{(r,s)}) > d \) for \( r \neq s, 1 \leq r, s \leq k \). Thus the graph \( B(k, m) = G \) and then the graphs \( B(k, m) \) for \( k, m > N + 2 \) cannot be constructed from \( \Psi \) by extensions \( \Xi \). Hence the class of \( \omega_k \)-graphs, \( d \geq 4 \) cannot be constructed from \( \Psi \) by extensions \( \Xi \). This proof also proves the same assertion for \( \omega_k \)-graphs, \( e \)-critical and \( v \)-critical graphs of diameter \( d \geq 4 \), since we used the property of diameter of \( B \) only. This completes the proof.

The Moore graphs can be defined as a graph of diameter \( d \geq 2 \) and girth \( 2d + 1 \), see [1]. So the Moore graphs of diameter two are \( \omega_2 \)-graphs. Three such graphs are known and the existence of Moore graphs of diameter two and degree 57 is possible.

The existence of regular graphs of diameter \( d \geq 2 \) and girth \( 2d \) is studied in [2]. We only note that the next corollary follows from second part of the above proof.

**Corollary 1.** Let \( \Psi \) and \( \Xi \) be finite sets of graphs. Then the regular graphs of diameter three and of girth six cannot be constructed from the set \( \Psi \) by extensions \( \Xi \).

For constructive description of discussed classes of graphs it is necessary to study other type of operations, too.

**REFERENCES**


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