Miroslav Katětov
Remarks on characters and pseudocharacters

Commentationes Mathematicae Universitatis Carolinae, Vol. 1 (1960), No. 1, 20–25

Persistent URL: http://dml.cz/dmlcz/104862

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1960

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized
documents strictly for personal use. Each copy of any part of this document must contain these
Terms of use.

This document has been digitized, optimized for electronic delivery and
stamped with digital signature within the project DML-CZ: The Czech Digital
Mathematics Library http://dml.cz
REMARKS ON CHARACTERS AND PSEUDOCHARACTERS  
Miroslav KATĚTOV, Praha  

The character $\chi(M, S)$ of a set $M$ in a topological space $S$ is defined as the least cardinal of a base around $M$; for the definition of the pseudocharacter $\psi(M, S)$, a pseudobase replaces a base. In a topological space $S$, a base (pseudobase) around a set $M \subseteq S$ is a collection $\mathcal{U}$ of neighborhoods of $M$ such that any neighborhood of $M$ contains some $U \in \mathcal{U}$ (the intersection of all $U \in \mathcal{U}$ is equal to $M$). These notions have been introduced, essentially by P. Alexandrov and P. Urysohn, Mémoire sur les espaces compacts, 1929. Various important results concerning the existence of spaces with prescribed characters and pseudocharacters of points are due to B. Pospíšil (e.g. Čas. pěst. mat. fys. 67 (1938), 249-255). Little seems to be known concerning characters of sets; from the few known results, recall the following one (J. Novák, Čas. pěst. mat. fys. 66 (1937), 206-209): if $S$ is metrizable, $M \subseteq S$, then $\chi(M, S) \leq \aleph_0$ if and only if $M - \text{Int} M$ is compact.

The present remarks, arisen in connection with the problem (considered by J. Novák) of the equality $\chi = \psi$ for the set of rational numbers (in the real line), contain several simple results concerning characters and pseudocharacters of sets as well as some related notions. It is to be noted that the equality $\mathcal{N} = 2^{\aleph_0}$ is not assumed. We consider completely regular topological spaces (called simply "spaces") only. The terminology of J. Kelley, General Topology, 1955, is used (with slight differences). The prover of a set $M$ is denoted $\text{card } M$; the letter $S$ always denotes a space.

I.

1.1. Definition. A $\mathcal{K}$-base (a $\mathcal{K}$-pseudobase) of is a collection $\mathcal{U}$ of compact subsets such that every compact
\[ K \subset S \quad \text{(respectively, every} \ x \in S \text{)} \] is contained in some \( A \in \mathcal{U} \). The \( k \)-character of \( S \), denoted \( \kappa \chi(S) \) (respectively, \( k \)-pseudocharacter, denoted \( \kappa \psi(S) \)) is the least cardinal of a \( k \)-base (\( k \)-pseudobase) of \( S \). Clearly, every \( k \)-base of \( S \) contains a \( k \)-base \( \mathcal{U} \) with card \( \mathcal{U} = \kappa \chi(S) \) and a \( k \)-pseudobase \( \mathcal{L} \) with card \( \mathcal{L} = \kappa \psi(S) \).

1.2. If \( S \) is compact, \( A \cup B = S \), \( A \cap B = \emptyset \), then \( \chi(A, S) = \kappa \chi(B) \), \( \psi(A, S) = \kappa \psi(B) \).

1.3. If \( S_1, S_2 \) are spaces, and \( \varphi \) is a continuous mapping of \( S_1 \) onto \( S_2 \) such that \( \varphi^{-1}(K) \) is compact whenever \( K \subset S_2 \) is so, then \( \kappa \chi(S_1) = \kappa \chi(S_2) \), \( \kappa \psi(S_1) = \kappa \psi(S_2) \).

1.4. Theorem. Let \( S_1, S_2 \) be locally compact, \( M_1 \subset S_1, M_2 \subset S_2 \), \( \overline{M_1} = S_1, \overline{M_2} = S_2 \); let \( M_1, M_2 \) be homeomorphic. Then \( \chi(M_1, S_1) = \chi(M_2, S_2) \), \( \psi(M_1, S_1) = \psi(M_2, S_2) \).

Proof. Consider only \( \chi \), the proof for \( \psi \) being quite analogous. Suppose first that \( S_1 \) is compact. Let \( f \) be a continuous mapping of the Čech–Stone compactification \( \beta M_1 \) onto \( S_1 \), \( f(x) = x \) for \( x \in M_1 \). Then \( f(\beta M_1 - M_1) = S_1 - M_1 \), and the restriction \( \varphi \) of \( f \) to \( \beta M_1 - M_1 \) satisfies the conditions from 1.3. Hence \( \kappa \chi(\beta M_1 - M_1) = \kappa \chi(S_1 - M_1) \) and therefore, by 1.2, \( \chi(M_1, \beta M_1) = \chi(M_1, S_1) \). This implies the validity of the theorem for compact \( S_1, S_2 \). If \( S_i \) are locally compact, choose compact \( T_i \) of \( S_i \) with \( \overline{S_i} = T_i \). Then \( S_i \) are open in \( T_i \) and therefore \( \chi(M_i, S_i) = \chi(M_i, T_i) \) from which the theorem follows.

1.5. By 1.4, for a given \( S \), the cardinals \( \chi(S, K), \psi(S, K) \) where \( S \subset K \), \( \overline{S} = K \), \( K \) is compact, do not depend on \( K \); they will be denoted \( \epsilon \chi(S), \epsilon \psi(S) \) and called external character (pseudocharacter) of \( S \).

Two spaces \( S_1, S_2 \) will be called associated if the these is a compact space \( K \) and subspaces \( S_i \subset K \) homeomorphic with \( S_i \) such that \( S_1' \cup S_2' = K \), \( S_1' \cap S_2' = \emptyset \).
Clearly, if \( S_1 \) and \( S_2 \) are associated, then 
\[
\varepsilon \chi (S_1) = \varepsilon \chi (S_2) , \quad 
\varepsilon \psi (S_1) = \varepsilon \psi (S_2) .
\]

Clearly, \( \chi (S_1, R) \leq \varepsilon \chi (S) \) if \( S \) is dense in the space \( R \); if not, it may happen e.g. that 
\[
\chi (S_1, R) > \aleph_0 , \quad \varepsilon \chi (S) = 1 , \quad \chi (S_1, R) = \aleph_0 .
\]

1.6. If \( S \) is locally compact \( \sigma \)-compact, then 
\[
\varepsilon \chi (S) \leq \aleph_0 .
\]

1.7. If \( \varepsilon \chi (S) \leq \aleph_0 \), and \( \chi (x, S) \leq \aleph_0 \) for every \( x \in S \), then \( S \) is locally compact \( \sigma \)-compact.

Proof. Suppose that \( S \) is not locally compact at \( a \in S \). Let \( A_n , n = 1, 2, \ldots \), form a \( \mathcal{B} \)-base of \( S \); let \( G_n \) form a base around \( a \) and let \( G_1 \supset G_2 \supset \ldots \).

Since \( G_n - A_n \neq \emptyset \), choose \( x_n \in G_n - A_n \), 
\( x_n \to a \); denote \( K \) the set consisting of \( a \) and all \( x_n \).
Then \( K \) is compact, \( K - A_n \neq \emptyset , \ n = 1, 2, \ldots \), 
which is a contradiction.

Remark. It is easy to see that the assumption \( \chi (x, S) \leq \aleph_0 \) cannot be omitted.

2.1. If \( R \) is an ordered set, let the least cardinal of 
a cofinal set in \( R \) be called cofinality character of \( R \).

Let \( \mathcal{N}^N \) denote the set of all sequences of natural numbers 
ordered as follows: \( \{ f_n \} \leq \{ g_n \} \) if 
(and only if) \( f_n \leq g_n \) for every \( n \). The 
cofinality character of \( \mathcal{N}^N \) will be denoted \( \mathcal{b} \).

It is clear that \( \chi_1 \leq \mathcal{b} \leq 2^{\aleph_0} ; \) by the 
author's knowledge neither of the equalities \( \chi_1 = \mathcal{b} \), \( \mathcal{b} = 2^{\aleph_0} \) 
has been proved as yet (nor disproved, of course).

Order the set \( F \) of all sequences of positive numbers as 
follows: \( \{ f_n \} \) precedes \( \{ g_n \} \) if (and only if) 

2.2. If $S$ is metrizable, $M \subset S$ is $\sigma$-compact, then $\chi(M, S) = \mathcal{B}$.

Proof. Let $M = \bigcup_{n=1}^{\infty} K_n$, $K_n$ compact. Let $A$ be cofinal in $N^N$. Choosing a metric $\rho$ for $S$, put $G_n, \xi = \{x \in S : \rho(x, K_n) < \frac{1}{n}\}$, and, for any $\xi = \{\xi_n\} \in N^N$, $U_\xi = \bigcup_{n=1}^{\infty} G_n, \xi_n$.

If $H$ is a neighborhood of $M$, choose $K_n$ with $G_n, \xi_n \subset H$ and $x \in A$ with $\{\xi_n\} \subseteq \xi$; then $M \subset U_\xi \subset H$. Hence $U_\xi, x \in A$, form a base around $M$.

2.3. Let $S$ be metrizable, $M \subset S$. If $M - \text{Int} M$ is not compact, then $\chi(M, S) = \mathcal{B}$.

Proof. There exist (distinct) points $b_n \in M - \text{Int} M$ such that $\{b_n\}$ has no cluster point in $M$. Choose a metric $\rho$ for $S$ and put, for any neighborhood $G$ of $M$, $\varphi(G) = \{\rho(b_n, S - G)\} \in F$ (see 2.1). Let $\mathcal{U}$ be a base around $M$. If $\{\xi_n\} \in F$, choose $x_n \in S - M$ with $\rho(x_n, b_n) < \min \left(\frac{1}{n}, \xi_n\right)$. Since $H = S - \cup(x_n)$ is a neighborhood of $M$, there is $U \in \mathcal{U}$ with $U \subset H$. Since $\rho(-b_n, S - U) \subseteq \rho(b_n, x_n), \{\xi_n\}$ precedes $\rho(U)$ in $F$.

Thus $\rho(U), U \in \mathcal{U}$, form a cofinal set in $F$.

2.4. Theorem. Let $S$ be metrizable; let $M \subset S$ be $\sigma$-compact. Then $\chi(M, S) = \mathcal{B}$ if and only if $M - \text{Int} M$ is not compact.

Remark. For instance, in $E_n$ the character of every non-compact closed set (different from $E_n$) is $\mathcal{B}$.

3.

3.1. Definition. A space $S$ will be called a $\lambda$-space if there is a transitive relation $\sigma$ on $S$ and a set $A$ such that the sets $\{x \in S : x \sigma a\}, a \in A$, form a
Clearly, any well ordered space is a \( \lambda \)-space.

Remark. It is easy to prove that \( S \) is a \( \lambda \)-space if and only if it satisfies one of the following equivalent conditions: (a) there is a \( \mathcal{K} \)-base \( \mathcal{A} \) such that, for any \( A \in \mathcal{A} \), \( A - \bigcup_{\lambda \in A} \lambda = \emptyset \).

(b) there is a \( \mathcal{K} \)-pseudobase \( \mathcal{A} \) and a mapping \( \psi \) of the system \( \mathcal{K} \) of all compact \( K \subseteq S \) into \( S \) such that \( K \in \mathcal{K} , A \in \mathcal{A} , \psi(K) \in A \) implies \( K \subseteq A \).

3.2. Theorem. If \( S \) is a \( \lambda \)-space, then \( \mathcal{K}(S) = \mathcal{K} \psi(S) \).

Proof. Let \( \sigma , A \) be as in 3.1. Clearly, there is \( B \subseteq A \) with \( \text{card} B = \mathcal{K} \psi(S) \) such that the system \( B \) of all \( \{ x \in S : x \sigma A \} \), \( A \in B \) is a \( \mathcal{K} \)-pseudobase. It is easy to prove that \( B \) is also a \( \mathcal{K} \)-base.

3.3. Let \( \mathcal{A} \) be a system of compact sets \( A \subseteq S \) such that (1) for any compact \( K \subseteq S \), \( K \in \bigcup_{\lambda \in A} \lambda \), for some \( \lambda = \lambda(A) \).

Then \( \mathcal{A}' = \mathcal{A} \). Then \( S \) is a \( \lambda \)-space.

Proof. By (2), we can choose, for any \( A \in \mathcal{A} \), a point \( \lambda(A) \in A \) contained in no \( x \in A \), \( x \neq A \).

Let \( \mathcal{A} \) be directed by a relation \( \subseteq \) in such a way that all \( \{ x \in \mathcal{A} : x \subseteq A \} \), \( A \in \mathcal{A} \), are finite. For \( A \in \mathcal{A} \), \( \lambda \in S \), put \( x \sigma y \) if (and only if) there are \( A_1 \in \mathcal{A} \), \( A_2 \in \mathcal{A} \) with \( x \in A_1 \), \( A_1 \subseteq A_2 \), \( y = \lambda(A_2) \). It is easy to see that \( \sigma \) is transitive. If \( \lambda = \lambda(A) \), then \( \{ x \in S : x \sigma y \} \) is equal to \( \bigcup_{\lambda \in A} \lambda \), hence compact. Condition (1) implies that \( \{ x \in S : x \sigma y \} \) (since \( \mathcal{A} \) is directed) that \( \{ x \in S : x \sigma y \} \).
3.4. The cartesian product of \( \mathcal{A} \)-spaces is a \( \mathcal{A} \)-space.

Proof. Let \( S_\xi, \xi \in \mathcal{Z} \), be \( \mathcal{A} \)-spaces, \( S = \prod S_\xi \). Let \( A_\xi, A_\xi \) be (for \( S_\xi \)) as in 3.1. Put \( \{ x_\xi \} \subset \{ y_\xi \} \) if (and only if) \( x_\xi \rho y_\xi \) for every \( \xi \); put \( A = \prod A_\xi \). Then \( A, \xi \) possess (for \( S \)) properties required in 3.1.

3.5. Theorem. The cartesian product of locally compact paracompact spaces is a \( \mathcal{A} \)-space.

Proof. Let \( S \) be locally compact paracompact. Then there is a locally finite open cover \( \{ U_\alpha \} \) such that \( \overline{U_\alpha} \) are compact. Clearly, there exists a subcover \( \{ U_\beta \} \) and points \( x_\beta \in U_\beta \) such that no \( x_\beta \) lies in \( U_\beta' \), \( \beta \neq \beta' \). By a well known theorem, there exist open \( V_\beta \) with \( x_\beta \rho V_\beta \), \( V_\beta \subset U_\beta \), \( UV_\beta = S \). The collection of all \( V_\beta \) has properties indicated in 3.3; hence \( S \) is a \( \mathcal{A} \)-space. Now apply 3.4.

Remark. It is easy to see that \( \mathcal{K} X(S) = \mathcal{K} Y(S) \) for any locally compact \( S \); nevertheless, I do not know whether \( \mathcal{K} X(S) = \mathcal{K} Y(S) \) holds whenever \( S \) is a product of locally compact spaces.

3.6. Corollary. Let \( R \) denote the space of rational numbers, \( J \) that of irrational ones. Then \( \varepsilon X(R) = \varepsilon Y(R) = \mathcal{K} X(J) = \mathcal{K} Y(J) = \mathfrak{K}_0 \).

Proof. By 2.4, \( \varepsilon X(R) = \mathfrak{K}_0 \); hence, \( R \) and \( J \) being associated, \( \mathcal{K} X(J) = \mathfrak{K}_0 \). Since \( J \) is homeomorphic to the product of \( \mathfrak{K}_0 \) discrete countable spaces, we have, by 3.5, \( \mathcal{K} Y(J) = \mathfrak{K}_0 \), hence \( \varepsilon Y(R) = \mathfrak{K}_0 \).

Remark. The conjecture seems probable that \( \mathcal{K} X(R) = \varepsilon X(J) = \mathfrak{K}_0 \).