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Remarks on characters and pseudocharacters

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respectively, every \( x \in S \) is contained in some \( A \in \mathcal{U} \). The \( k \)-character of \( S \), denoted \( \kappa \chi(S) \) (respectively, \( k \)-pseudocharacter, denoted \( \kappa \psi(S) \)) is the least cardinal of a \( k \)-base (\( k \)-pseudobase) of \( S \).

Clearly, every \( k \)-base of \( S \) contains a \( k \)-base \( \mathcal{U} \) with card \( \mathcal{U} = \kappa \chi(S) \) and a \( k \)-pseudobase \( \mathcal{B} \) with card \( \mathcal{B} = \kappa \psi(S) \).

1.2. If \( S \) is compact, \( A \cup B = S \), \( A \cap B = \emptyset \), then \( \chi(A, S) = \kappa \chi(B) \), \( \psi(A, S) = \kappa \psi(B) \).

1.3. If \( S_1, S_2 \) are spaces, and \( \mathcal{G} \) is a continuous mapping of \( S_1 \) onto \( S_2 \) such that \( \mathcal{G}^{-1}(K) \) is compact whenever \( K \subset S_2 \), then \( \kappa \chi(S_1) = \kappa \chi(S_2) \), \( \kappa \psi(S_1) = \kappa \psi(S_2) \).

1.4. Theorem. Let \( S_1, S_2 \) be locally compact, \( M_1 \subset S_1 \), \( \overline{M_1} = S_1 \), \( M_2 \subset S_2 \), \( \overline{M_2} = S_2 \); let \( M_1, M_2 \) be homeomorphic. Then \( \chi(M_1, S_1) = \chi(M_2, S_2) \), \( \psi(M_1, S_1) = \psi(M_2, S_2) \).

Proof. Consider only \( \chi \), the proof for \( \psi \) being quite analogous. Suppose first that \( S_1 \) is compact. Let \( f \) be a continuous mapping of the Čech–Stone compactification \( \beta M_1 \) onto \( S_1 \), \( f(x) = x \) for \( x \in M_1 \). Then \( f(\beta M_1 - M_1) = S_1 - M_1 \), and the restriction \( \mathcal{G} \) of \( f \) to \( \beta M_1 - M_1 \) satisfies the conditions from 1.3. Hence \( \kappa \chi(\beta M_1 - M_1) = \kappa \chi(S_1 - M_1) \) and therefore, by 1.2, \( \chi(M_1, \beta M_1) = \kappa \chi(S_1, S_1) \). This implies the validity of the theorem for compact \( S_1, S_2 \). If \( S_i \) are locally compact, choose compact \( T_i \supset S_i \) with \( \overline{T_i} = T_i \). Then \( S_i \) are open in \( T_i \) and therefore \( \chi(M_i, S_i) = \chi(M_i, T_i) \) from which the theorem follows.

1.5. By 1.4, for a given \( S \), the cardinals \( \chi(S, K) \), \( \psi(S, K) \) where \( S \subset K \), \( \overline{S} = K \), \( K \) is compact, do not depend on \( K \); they will be denoted \( e \chi(S) \), \( e \psi(S) \) and called external character (pseudocharacter) of \( S \).

Two spaces \( S_1 \), \( S_2 \) will be called associated if the these is a compact space \( K \) and subspaces \( S_i \subset K \) homeomorphic with \( S_i \) such that \( S_1 \cup S_2 = K \), \( S_1 \cap S_2 = \emptyset \).
Clearly, if \( S_1 \), \( S_2 \) are associated, then 
\[
\varepsilon \chi (S_1) = \varepsilon \chi (S_2),
\]
\[
\varepsilon \psi (S_1) = \varepsilon \psi (S_2).
\]

Clearly, \( \chi (S, R) \leq \varepsilon \chi (S) \) if \( S \) is dense in the space \( R \); if not, it may happen e.g. that 
\[
\chi (S, R) > \aleph_0, \quad \varepsilon \chi (S) = 1,
\]
\[
\chi (S, R) = \aleph_0.
\]

1.6. If \( S \) is locally compact \( \sigma \)-compact, then 
\[
\varepsilon \chi (S) \leq \aleph_0.
\]

1.7. If \( \varepsilon \chi (S) \leq \aleph_0 \), and \( \chi (x, S) \leq \aleph_0 \) for every \( x \in S \), then \( S \) is locally compact \( \sigma \)-compact.

Proof. Suppose that \( S \) is not locally compact at \( a \in S \).
Let \( A_n, n = 1, 2, \ldots \), form a base of \( S \); let
\( G_n \) form a base around \( a \) and let \( G_1 \supset G_2 \supset \ldots \).
Since \( G_n - A_n \neq \emptyset \), choose \( x_n \in G_n - A_n \),
\( x_n \to a \); denote \( K \) the set consisting of \( a \) and all \( x_n \).
Then \( K \) is compact, \( K - A_n \neq \emptyset \), \( n = 1, 2, \ldots \),
which is a contradiction.

Remark. It is easy to see that the assumption \( \chi (x, S) \leq \aleph_0 \) cannot be omitted.

2.1. If \( R \) is an ordered set, let the least cardinal of a cofinal set in \( R \) be called cofinality character of \( R \).
Let \( N^N \) denote the set of all sequences of natural numbers ordered as follows: \( \{ \xi_n \} \leq \{ \eta_n \} \) if
( and only if ) \( \xi_n \leq \eta_n \) for every \( n \). The cofinality character of \( N^N \) will be denoted \( \mathcal{C} \).

It is clear that \( \chi_1 \leq \mathcal{C} \leq 2^{\aleph_0} \); by the author's knowledge neither of the equalities \( \chi_1 = \mathcal{C}, \mathcal{C} = 2^{\aleph_0} \)
has been proved as yet (nor disproved, of course).

Order the set \( F \) of all sequences of positive numbers as follows: \( \{ \xi_n \} \) precedes \( \{ \eta_n \} \) if (and only if)
2.2. If $S$ is metrizable, $M \subset S$ is $\sigma$-compact, then $\chi(M, S) \leq \mathcal{B}$.

Proof. Let $M = \bigcup_{n=1}^{\infty} K_n$, $K_n$ compact. Let $A$ be cofinal in $N^N$. Choosing a metric $\rho$ for $S$, put $G_n, \xi = \{x \in S : \rho(x, K_n) < \frac{1}{n}\}$, and, for any $x = \{\xi_n\} \in N^N$, $U_x = \bigcup_{n=1}^{\infty} G_n, \xi_n$.

If $H$ is a neighborhood of $M$, choose $K_n$ with $G_n, \xi_n \subset H$ and $x \in A$ with $\{\xi_n\} \subset x$; then $M \subset U_x \subset H$. Hence $U_x$, $x \in A$, form a base around $M$.

2.3. Let $S$ be metrizable, $M \subset S$. If $M - \text{Int} M$ is not compact, then $\chi(M, S) \geq \mathcal{B}$.

Proof. There exist (distinct) points $\xi_n \in M - \text{Int} M$ such that $\{\xi_n\}$ has no cluster point in $M$. Choose a metric $\rho$ for $S$ and put, for any neighborhood $G$ of $M$, $\varphi(G) = \{\rho(\xi_n, S - G)\} \in F$ (see 2.1). Let $U$ be a base around $M$. If $\{\xi_n\} \in F$, choose $x_n \in S - M$ with $\rho(x_n, \xi_n) < \min \left(\frac{1}{n}, \xi_n\right)$. Since $H = S - \bigcup(x_n)$ is a neighborhood of $M$, there is $U \in U$ with $U \subset H$. Since $\rho(-\xi_n, S - U) \leq \rho(\xi_n, x_n)$, $\{\xi_n\}$ precedes $\varphi(U)$ in $F$.

Thus $\varphi(U)$; $U \in U$, form a cofinal set in $F$.

2.4. Theorem. Let $S$ be metrizable; let $M \subset S$ be $\sigma$-compact. Then $\chi(M, S) = \mathcal{B}$ if and only if $M - \text{Int} M$ is not compact.

Remark. For instance, in $E^n$ the character of every non-compact closed set (different from $E^n$) is $\mathcal{B}$.

3.

3.1. Definition. A space $S$ will be called a $\lambda$-space if there is a transitive relation $\sigma$ on $S$ and a set $A$ such that the sets $\{x \in S : x \sigma a\} \subset a \in A$, form a
3.4. The cartesian product of \( \mathcal{A} \) -spaces is a \( \mathcal{A} \) -space.

Proof. Let \( S^\xi, \xi \in Z \), be \( \mathcal{A} \) -spaces, \( S = \prod S^\xi \).
Let \( \sigma^\xi, A^\xi \) be \( \mathcal{A} \) -spaces as in 3.1. Put \( \{ x^\xi \} \sigma \in \{ y^\xi \} \) if (and only if) \( x^\xi \sigma y^\xi \) for every \( \xi \); put \( A = \prod A^\xi \). Then \( A \), \( \xi \) possess properties required in 3.1.

3.5. Theorem. The cartesian product of locally compact para-compact spaces is a \( \mathcal{A} \) -space.

Proof. Let \( S \) be locally compact para-compact. Then there is a locally finite open cover \( \{ U_\alpha \} \) such that \( \overline{U_\alpha} \) are compact. Clearly, there exists a subcover \( \{ V_\beta \} \) such that no \( \alpha \in \beta \) lies in \( U_\beta' \), \( \beta \neq \beta' \). By a well known theorem, there exist open \( V_\beta \) with \( \alpha \in V_\beta' \), \( V_\beta \subset U_\beta \), \( U V_\beta = S \). The collection of all \( V_\beta \) has properties indicated in 3.3; hence \( S \) is a \( \mathcal{A} \) -space. Now apply 3.4.

Remark. It is easy to see that \( k_X(S) = k_Y(S) \) for any locally compact \( S \); nevertheless, I do not know whether \( k_X(S) = k_Y(S) \) holds whenever \( S \) is a product of locally compact spaces.

3.6. Corollary. Let \( \mathbb{R} \) denote the space of rational numbers, \( J \) that of irrational ones. Then \( k_X(\mathbb{R}) = k_Y(\mathbb{R}) = \mathcal{K} \).

Proof. By 3.4, \( k_X(\mathbb{R}) = \mathcal{K} \); hence, \( \mathbb{R} \) and \( J \) being associated, \( k_X(J) = \mathcal{K} \). Since \( J \) is homeomorphic to the product of \( \aleph_0 \) discrete countable spaces, we have, by 3.5, \( k_Y(J) = \mathcal{K} \), hence \( k_Y(\mathbb{R}) = \mathcal{K} \).

Remark. The conjecture seems probable that \( k_X(\mathbb{R}) = k_Y(J) = \mathcal{K} \).