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ON POINTWISE CONVERGENCE OF SEQUENCES OF CONTINUOUS FUNCTIONS

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If P is a topological space, we denote by $C(P)$ the set of all continuous real-valued functions on P . For $f, f_n \in C(P)$ ($n = 1, 2, \dots$) we define $f_n \xrightarrow{n} f$ if

and only if $f_n(x) \xrightarrow{n} f(x)$ for every $x \in P$

(if $f, f_n \in C(P)$ and $f_n \xrightarrow{n} f$ does not hold,

$f_n \xrightarrow{n} f$ is written). This convergence defines a topology μ on $C(P)$ in the well-known manner (for $A \subset C(P)$ μA consists of all $f \in C(P)$ such that $f_n \rightarrow f$ for some $f_n \in A$). Let us point out that, following

E. Čech [1], a topology μ on a set Q is defined as a mapping μ which to every $M \subset Q$ assigns a set $\mu M \subset Q$ and satisfies the following axioms: $\mu \emptyset = \emptyset$, $\mu(x) = (x)$, $\mu(M_1 \cup M_2) = \mu M_1 \cup \mu M_2$.

The condition $\mu(\mu M) = \mu M$, called axiom F by E. Čech, is not required in general; if it is satisfied, then μ is called an F -topology and (Q, μ) an F -space. For any topology μ on Q two further topologies are defined: $\tilde{\mu}$, the F -reduction of μ , which possesses an open base consisting of all $Q - \mu A$, $A \subset Q$; μ^* , the F -modification of μ , which is the finest of all F -topologies coarser than μ . Clearly $\tilde{\mu} = \mu$ or $\mu = \mu^*$ if and only if μ is an F -topology.

It is known that for some not at all exceptional spaces P , the space $(C(P), \mu)$ does not satisfy the axiom F ; therefore it will be interesting to consider the spaces $(C(P), \tilde{\mu})$ and $(C(P), \mu^*)$. In this note it is shown that for the most important spaces P , $(C(P), \tilde{\mu})$ is a discrete space (part II). In part I, the regularity of the space $(C(P), \mu^*)$ is studied.

The usual notation and terminology is used (with the above differences). The same symbol will be used for a sequence or double sequence and for the set of its members. The power of a set A will be denoted $\text{card } A$.

I would like to take this opportunity to thank Prof. M. Kařtřov for his useful hints, namely concerning theorem 2 in part II.

I.

We denote by N the set of all positive integers. If X is a set, X^N denotes the collection of all sequences $\{x_m\}_{m=1}^{\infty}$, $x_m \in X$.

Definition : Let $\alpha = \{\alpha_m\} \in N^N$, $\beta = \{\beta_m\} \in N^N$. We write $\alpha \succ \beta$ if there exists an m_0 such that $\alpha_m > \beta_m$ for all $m \geq m_0$. If $\alpha \succ \beta$ does not hold, we write $\alpha \not\succ \beta$.

Proposition 1 : There exists a set $\mathcal{A} \subset N^N$ such that :

- (1) If $\alpha, \beta \in \mathcal{A}$, $\alpha \neq \beta$, then either $\alpha \succ \beta$ or $\beta \succ \alpha$.
- (2) If $\gamma \in N^N$, then there exists an $\alpha \in \mathcal{A}$ such that $\gamma \not\succ \alpha$.
- (3) If $\alpha^i \in \mathcal{A}$ ($i = 1, 2, \dots$), then there exists an $\alpha \in \mathcal{A}$ such that $\alpha \succ \alpha^i$ ($i = 1, 2, \dots$).

Proof : The existence of a set \mathcal{A} with properties (1), (2) follows from Zorn's lemma; (3) follows from (1) and (2) by means of the diagonal method.

Theorem 1 : If \mathcal{P} is a normal space which contains a discrete ¹⁾ family of power $\geq 2^{\aleph_0}$ consisting of open sets, then there exists a closed set $F \subset (C(\mathcal{P}), \mu)$ and a double sequence $A = \{f_{m_i, m_j}\} \subset C(\mathcal{P})$ such that :

- 1) $0 \notin F$ (0 denotes the function identically zero on \mathcal{P});

- 1) That is, locally finite and such that the closures of any two distinct sets of the family are disjoint.

- 2) $f_{m,n} \xrightarrow{n} 0$ for all $m \in N$;
 3) If U is a neighborhood of 0 in $(C(P), u)$,
 then $F \cap u(A \cap U) \neq \emptyset$.

Proof : Let $\{G_g \mid g \in \mathcal{G}\}$ be a discrete family of open subsets of P , $g \in G_g$, $\text{card } \mathcal{G} \geq 2^{\aleph_0}$.
 Let $\mathcal{A} \subset N^N$ possess properties (1), (2), (3) from proposition 1. Let φ be a one-to-one mapping of \mathcal{A} into \mathcal{G} .
 If $\beta \in \mathcal{A}$, we denote $\mathcal{R}_\beta = \{\alpha \in \mathcal{A} \mid \alpha \geq \beta\}$,
 $Z_\beta = \varphi(\mathcal{R}_\beta)$, $F_\beta = \{f \in C(P) \mid f|Z_\beta \geq 1\}$,
 $F = \bigcup_{\beta \in \mathcal{A}} F_\beta$. Property (3) of \mathcal{A} implies that F

is a closed set. For $g \in \mathcal{G}$ let $f^g \in C(P)$ be such that $f^g(g) = 1$, $f^g|P - G_g = 0$. We put, for $m, n \in N$ $A_{m,n} = \{\alpha = \{\alpha_k\} \in \mathcal{A} \mid \alpha_m \geq n\}$,
 $G_{m,n} = \varphi(A_{m,n})$, $f_{m,n} = \sum_{g \in G_{m,n}} f^g$, $A = \{f_{m,n}\}$.

It is easy to prove that $f_{m,n} \xrightarrow{n} 0$ for all $m \in N$.

If U is a neighborhood of 0 , then there exists a $\mathcal{X} = \{\alpha_m\} \in N^N$ such that $f_{m,n} \in U$ for $n \geq \alpha_m$. By property (2) of \mathcal{A} there exists a $\beta \in \mathcal{A}$, $\beta = \{\beta_m\}$, such that $\mathcal{X} \not\geq \beta$.
 Let $\{j_m\} \in N^N$ be an increasing sequence such that $\beta_{j_m} \geq \alpha_{j_m}$; put $h_m = f_{j_m, \beta_{j_m}}$,
 $f = \sum_{g \in \mathcal{G}} f^g$. It may be proved that $h_m \xrightarrow{m} f$,
 $f \in F$.

Corollary : If P is a normal space and there exists a discrete family of power $\geq 2^{\aleph_0}$ consisting of open subsets of P , then the space $(C(P), u^*)$ is not regular

Remark : If P contains a dense countable subset, then for every $A = \{f_{m,n}\} \subset C(P)$ such that $f_{m,n} \xrightarrow{n} 0$

for all $m \in N$, there exists a $\{\alpha_m\} \in N^N$ such that the set $B = \{f_{m,n} \in A \mid n \geq \alpha_m\}$ has, in

$(C(P), u)$ precisely one cluster point, the point 0 .

Problem: Let P contain a countable dense subset. Under what conditions is $(C(P), u^*)$ regular? (Clearly it is such if P itself is countable).

II.

In this part we shall examine the F -reduction of the topology u on $C(P)$. We notice that a space $(C(P), \tilde{u})$ is discrete if and only if for every $f \in C(P)$ there exists a $H \subset C(P)$ such that

1) for every $g \in C(P)$, $g \neq f$, there exists a $\{g_n\} \in H^N$ such that $g_n \xrightarrow{n} g$,

2) if $\{f_n\} \in H^N$, then $f_n \xrightarrow{n} f$.

Definitions:

We shall call $\sigma(P)$ every double sequence $\{f_{m,n}\}$ of functions from $C(P)$ such that

1) $f_{m,n} \xrightarrow{n} 0$ for all $m \in N$,

2) if $\{k_m\} \in N^N$ is increasing, $\{n_m\} \in N^N$, then $f_{k_m, n_m} \xrightarrow{m} 0$.

We shall call $j(P)$ every double sequence $\{h_{m,n}\}$ of functions from $C(P)$ such that:

1) $h_{m,n} \xrightarrow{n} 1$ for all $m \in N$

2) if $\{k_m\} \in N^N$ is increasing, $\{n_m\} \in N^N$, $\{g_m\} \in C(P)^N$, then $g_{k_m} \cdot h_{k_m, n_m} \xrightarrow{m} 1$.

Theorem 1. For every space P containing a dense countable subset the following properties are equivalent:

1) $(C(P), u)$ is not an F -space;

2) there exists an $\sigma(P)$;

3) there exists a $j(P)$;

4) $(C(P), \tilde{u})$ is discrete.

2) $\sigma(P)$ is a \mathcal{G} -system, as defined in [3].

Proof : No $(C(P), u)$ is discrete and therefore

4) \Rightarrow 1) is trivial.

1) \Leftrightarrow 2) is proved in [3].

2) \Rightarrow 3) : Let $\{f_{m,n}\} \subset C(P)$ be an $\sigma(P)$.

Let $\alpha_m \in C(E_1)$ be defined as follows :

$$\alpha_m(y) = 0 \quad \text{if } y \leq 1 - \frac{1}{m}$$

$$\alpha_m(y) = (m+1)(my - m + 1) \quad \text{if } 1 - \frac{1}{m} \leq y \leq 1 - \frac{1}{m+1}$$

$$\alpha_m(y) = 1 \quad \text{if } 1 - \frac{1}{m+1} \leq y.$$

For every $x \in P$ we put $h_{m,n}(x) = \alpha_m(1 - |f_{m,n}(x)|)$.

It is easy to prove that $\{h_{m,n}\}$ is a $j(P)$.

3) \Rightarrow 4) : Let $\{a_1, a_2, \dots\}$ be dense in P .

For $k, l \in N$ denote by $M_{k,l}$ the set of $f \in C(P)$ such that $|f(a_k) - 1| \geq \frac{1}{l}$;

clearly $\bigcup_{k,l} M_{k,l} = C(P) - (1)$. Let φ be

a one-to-one mapping of $N \times N$ onto N .

Let $\{h_{s,n}\}$ be a $j(P)$; since $h_{s,n} \xrightarrow{n} 1$,

we may suppose (replacing, if necessary, $\{h_{s,n}\}$ by $\{h_{s, n_s + n}\}$ with n_s sufficiently large)

that for any $k, l, n \in N$ and $s = \varphi(k, l)$, $|h_{s,n}(a_k) - 1| \leq \frac{1}{3l}$. Then, evidently,

$$(*) \quad |f(a_k) \cdot h_{s,n}(a_k) - 1| \geq \frac{1}{3l} \quad \text{if } s = \varphi(k, l), f \in M_{k,l}$$

Denote by H_1 the set of all $f \cdot h_{s,n}$ where $k \in N$, $l \in N$, $f \in M_{k,l}$, $s = \varphi(k, l)$. Clearly

$f \cdot h_{\varphi(k,l), n} \xrightarrow{n} f$ if $f \in M_{k,l}$; hence

$\cup H_1 \supset C(P) - (1)$. Suppose $g_i \in H_1$,

$g_i \xrightarrow{i} 1$. Then, for some $\{k_i\}, \{l_i\},$

$\{n_i\}$ from N^N , we have $g_i = f_i \cdot h_{s_i, n_i}$,

$f_i \in M_{k_i, l_i}$, $s_i = \varphi(k_i, l_i)$. Since $\{h_{s,n}\}$

is a $j(P)$ and $f_i \cdot h_{s_i, n_i} \xrightarrow{i} 1$, almost all s_i are equal to some $s = \varphi(k, l)$. Therefore, for large i , $f_i \in M_{h, l}$ and by $(*)$, $|f_i(a_k) \cdot h_{s_i, n_i}(a_k) - 1| \geq \frac{1}{3l}$; this contradicts $g_i \xrightarrow{i} 1$. Thus $1 \notin u H_1$. Putting, for $g \in C(P)$, $H_g = C(P) - (g)$, $u H_g = \{f - 1 + g \mid f \in H_1\}$ we obtain $u H_g = C(P) - (g)$.

There follow several criteria which show, in some cases, when $(C(P), u)$ is an F -space and when it is not so (and therefore, if P contains a countable dense subset, $(C(P), \tilde{u})$ is discrete). Proofs are omitted.

Proposition 1. (contained in [4]). If P is a countable space, then $(C(P), u)$ is an F -space (metrizable even),.

Proposition 2. If for every $f \in C(P)$ there exists a countable $A \subset P$ such that f is constant on $P - A$, then $(C(P), u)$ is an F -space.

Proposition 3. If P is a dense-in-itself non-meager ³⁾ normal space containing a countable metrizable dense subspace, then $(C(P), u)$ is not an F -space.

Proposition 4. If P is the β -compactification of some infinite discrete space, then $(C(P), u)$ is not an F -space.

Proposition 5. Let A be a closed G_δ subset of a normal space P . $(C(A), u)$ is not an F -space, then nor is $(C(P), u)$. If $(C(A), \tilde{u})$ is discrete and $P - A$ contains a countable dense (in $P - A$) subset, then $(C(P), \tilde{u})$ is discrete also.

If in theorem 1 we omit the condition of separability of P , the theorem ceases to hold. In this case only properties (1), (2), (3) are equivalent. Property (4) is not equivalent to them,

3) That is, P is not a union of countably many nowhere dense sets.

as following example shows.

Let A be some space such that $(C(A), \mu)$ is not an F space. Let M be a set, $\text{card } M > (\text{card } C(A))^{\aleph_0}$.

Let $B = M \cup \{\xi\}$ be a compact Hausdorff space with a unique non-isolated point ξ . We put $P = A \cup B$, $A \cap B = \emptyset$, A, B are closed in P . Clearly, if A is a compact Hausdorff space, P is such also.

$(C(P), \mu)$ is not an F -space (the existence of an $\sigma(P)$ is evident), but $(C(P), \tilde{\mu})$ is not discrete - this may be proved using proposition 2.

Nevertheless $(C(P), \tilde{\mu})$ is also discrete for some important non-separable spaces, e.g. for α -separable metric spaces with $\alpha = \alpha^{\aleph_0}$, and also for their β -compactifications, as shown in the following theorem.

In this theorem and its proof, \bar{A} denotes the closure of a set $A \subset P$ in P .

Theorem 2. If P is a normal space containing a discrete normally imbedded ⁴⁾ subset of power $\alpha = \alpha^{\aleph_0}$ and a dense subset of power $\leq 2^\alpha$, then $(C(P), \tilde{\mu})$ is discrete.

Lemma: Let Z, Ξ be sets, $\text{card } Z = \alpha = \alpha^{\aleph_0}$, $\text{card } \Xi = 2^\alpha$. Then there exists a collection $\{Z_{\xi, n} \mid \xi \in \Xi, n \in N\}$ of subsets of Z such that

(1) if $\xi \in \Xi$, $n_1, n_2 \in N$, $n_1 \neq n_2$, then

$$Z_{\xi, n_1} \cap Z_{\xi, n_2} = \emptyset;$$

4) A set Q is said to be normally imbedded in a space P if $Q \subset P$ and every bounded continuous function on Q can be extended continuously to P . By discrete subset we means, in this theorem, simply a subset which, as a subspace, contains isolated points only.

(2) if $\xi_i \in \Xi$ are distinct, $n_i \in N$, then $\bigcap_{i=1}^{\infty} Z_{\xi_i, n_i}$

is uncountable.

This lemma follows easily from the following proposition (cf. [1], p. 489). If R is the topological product of 2^α intervals $\langle 0, 1 \rangle$, then R contains a dense subset of power $\leq \alpha^{No}$ whose intersection with every non-void G_δ subset of R is uncountable.

(Let φ be a one-to-one mapping of A into Z . If $R = \{ \{n_\beta\} \mid n_\beta \in \langle 0, 1 \rangle, \beta \in \Xi \}$, $A_{\xi, m} = \{ \{n_\beta\} \in A \mid n_\beta \in (\frac{1}{m+1}, \frac{1}{m}) \}$, we put $Z_{\xi, m} = \varphi(A_{\xi, m})$).

Proof of theorem 2: Let $X \subset P$ be dense, $Z \subset P$ discrete normally imbedded, $\text{card } Z = \alpha = \alpha^{No}$, $\text{card } X \leq 2^\alpha$. Put $\Xi = X \times N$ and let

$\{ Z_{\xi, m} \mid \xi \in \Xi, m \in N \}$ be a collection of subsets of Z with properties (1), (2) from the lemma, and such that if $\xi = [x, m] \in \Xi$, then $x \notin Z_{\xi, m}$.

Since Z is normally imbedded, the collection $\{ \overline{Z_{\xi, m}} \}$ is disjoint, be (1), for any fixed $\xi \in \Xi$. This

implies, by normality of P , the existence of open sets $G_{\xi, m} \supset \overline{Z_{\xi, m}}$ such that, for any fixed $\xi = [x, m]$

$\{ G_{\xi, m} \}$ is disjoint, $x \in G_{\xi, m}$. Now choose, for any $\xi \in \Xi$, $m \in N$, a function $h_{\xi, m} \in C(P)$ equal to 0 on $Z_{\xi, m}$ and to 1 on $P - G_{\xi, m}$. Clearly

(a) for any $\xi \in \Xi$ $h_{\xi, m} \xrightarrow{m} 1$, (b) for any $\xi \in [x, m] \in \Xi$ $h_{\xi, m}(x) = 1$, (c) if $\xi_k \in \Xi$ are distinct, $n_k \in N$, $f_k \in C(P)$, then

$f_k: h_{\xi_k, n_k} \xrightarrow{k} 1$ (since $h_{\xi_k, n_k}(y) = 0$ where $y \in \bigcap_{k=1}^{\infty} Z_{\xi_k, n_k}$).

Let H_1 denote the set of all $f \cdot h_{\xi, m}$ with $\xi = [x, m]$, $f \in C(P)$, $|f(x) - 1| \geq \frac{1}{m}$. By (a), $f \cdot h_{\xi, m} \xrightarrow{m} f$; hence $\cup H_1 \supset C(P) - (1)$. Suppose that $g_k \in H_1$, $g_k \xrightarrow{k} 1$. Then $g_k = f_k \cdot h_{\xi_k, n_k}$, $\xi_k = [x_k, m_k]$, $f_k \in C(P)$,

$|f_k(x_k) - 1| \geq \frac{1}{m_k}$. Since $g_k \xrightarrow{k} 1$, we

obtain, by (c), that almost all f_k are equal to some $[x, m] \in \Xi$. Hence, for large k , $|f_k(x) - 1| \geq \frac{1}{m}$ and by (b) $g_{f_k, n_k}(x) = 1$; therefore

$|g_k(x) - 1| \geq \frac{1}{m}$ which is contradiction. We

have proved that $\mu H_1 = C(P) - (1)$. Now, if $g \in C(P)$, put $H_g = \{g + f - 1 \mid f \in H_1\}$; then $\mu H_g = C(P) - (g)$.

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