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ON CONVERGENCE OF SEQUENCES OF FUNCTIONS

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If \((Ω_{\Lambda}, \mathcal{A})\) \((\Lambda \in \Lambda)\) are spaces, we can define on a cartesian product \(Ω\) of sets \(Ω_{\Lambda}\) a convergence of sequences by a well known way: For \(x^{n} \in Ω_{\Lambda} \quad (n = 1, 2, \ldots), x^{n} \rightarrow x\) if and only if \(x_{j}^{n} \rightarrow x_{j}\) in the space \((Ω_{\Lambda}, \mathcal{A})\) for every \(j \in \Lambda\) (\(x_{j}\) denotes the \(j\)-th coordinate of the point \(x\)). This convergence defines a topology \(\mathcal{A}\) on \(Ω\) in the well known way (for \(A \subseteq Ω\) \(\mathcal{A}\) consists of all \(x \in Ω\) such that \(x^{n} \rightarrow x\) for some \(x_{n} \in A\). Following J. Novák [3], we call \((Ω, \mathcal{A})\) an \(L\)-product of spaces \((Ω_{\Lambda}, \mathcal{A})\) and denote \((Ω, \mathcal{A}) = \bigotimes_{\Lambda} (Ω_{\Lambda}, \mathcal{A})\).

Let us point out that, following E. Čech [1] a topology \(\mathcal{A}\) on the set \(Ω\) is defined as a mapping \(\mathcal{A}\), which to every \(M \subseteq Ω\) assigns a set \(\mu M \subseteq Ω\) and satisfies the following axioms: \(\mu \emptyset = \emptyset\), \(\mu (x) = \{x\}\), \(\mu (M_{1} \cup M_{2}) = \mu M_{1} \cup \mu M_{2}\). The condition \(\mu (\mu M) = \mu M\), called axiom \(F\) by E. Čech, is not required in general; if it is satisfied, then \(\mathcal{A}\) is called an \(F\)-topology and \((Ω, \mathcal{A})\) an \(F\)-space; if it does not hold, then \(\mathcal{A}\) is called a non-\(F\)-topology and \((Ω, \mathcal{A})\) a non-\(F\)-space.

For any topology \(\mathcal{A}\) on \(Ω\) two further \(F\)-topologies are defined: \(\mathcal{A}^{*}\), the \(F\)-reduction of \(\mathcal{A}\), which has an open base consisting of all \((\Lambda, \mathcal{A} A)\), \(A \subseteq Ω\); \(\mathcal{A}^{*}\), the \(F\)-modification of \(\mathcal{A}\), which is the finest of all \(F\)-topologies, coarser than \(\mathcal{A}\). Clearly \(\mathcal{A} = \mathcal{A}^{*}\) or \(\mathcal{A} = \mathcal{A}^{*}\) if and only if \(\mathcal{A}\) is an \(F\)-topology.

In this note an \(L\)-product of two-point spaces is studied. The smallest cardinal number \(\gamma\) is found, for which an \(L\)-product of \(\gamma\) two-point spaces is non-\(F\)-space, event. it is not countably compact.
It is shown, that the $\mathcal{L}$-product $\left(\mathbb{Q}, \mu\right)$ of uncountable number of two-point spaces and $\left(\mathbb{C}, \mu^*\right)$ are not regular. Several criteria are given, when the space $\left(\mathbb{C}, \tilde{\mu}\right)$ is discrete (I. - III.).

In IV.- VIII. similar questions for subspaces of the space of real-valued functions on some $F$-space are studied.

In the whole note proofs are omitted.

In this note, $\mathbb{N}$ denotes the set of all natural numbers. If $A, B$ are sets, $A^B$ denotes the set of all mappings of $B$ into $A$. If $x \in A^B$, $x \in B$ then the element, corresponding to $x$ in the mapping $\alpha$, is denoted $\alpha(x)$ or $\alpha_x$.

If $\sigma \in A^B$, $C \subseteq B$, then $\alpha|C$ denotes the mapping of $C$ into $A$, for which $\alpha|C(x) = \alpha(x)$ for all $x \in C$. $\aleph_0$ denotes an arbitrary cardinal number.

I. Countable compactness.

Definition:
The space $\left(\mathbb{C}, \mu\right)$ is called countably compact, if every infinite subset of $\mathbb{C}$ has a cluster point.

Theorem 1, 1:
Let $\left(\mathbb{C}, \mu\right)$ be a space. The following properties are equivalent:
1) $\left(\mathbb{C}, \mu\right)$ is countably compact.
2) $\left(\mathbb{C}, \mu\right)$ is countably compact.

Theorem 1, 1 does not hold for compactness only.

Definition.
Let $\sigma$ be the smallest power of a system $\mathcal{A}$ of subsets of $\mathbb{N}$, which has the following property:
if $S \subseteq \mathbb{N}$ is infinite, then there exists $A \in \mathcal{A}$ such that the sets $S \cap A$, $S - A$ are infinite.

Theorem 1, 2:
Let $\left(\mathbb{C}, \mu\right)$ be an $\mathcal{L}$-product of $\aleph_1$ two-point spaces.
The following properties are equivalent:
1) $\left(\mathbb{C}, \mu\right)$ is not countably compact.
2) $\aleph_1 \geq \sigma$. 

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II. F - axiom, order and regularity.
Definitions:
Let \( A \) be an infinite countable set, \( \alpha, \beta \in \mathbb{N}^A \).
We write \( \alpha \succeq \beta \) if \( \gamma(x) > \beta(x) \) for all \( x \in A \), except a finite number. If \( \alpha \succeq \beta \) does not hold, we write \( \alpha \succeq \beta \).

We say that \( \mathcal{A} \subset \mathbb{N}^A \) is an unbounded system in \( \mathbb{N}^A \) if for every \( \gamma \in \mathbb{N}^A \) there exists some \( \alpha \in \mathcal{A} \) such that \( \gamma \succeq \alpha \).

We say that \( \mathcal{A} = \{ \alpha^\lambda \} \subset \mathbb{N}^A \) is a hereditary unbounded system, if the system \( \{ \alpha^\lambda \mid \lambda \} \) is unbounded in \( \mathbb{N}^A \) for every infinite \( \lambda \in \mathbb{N} \).

We say that a set \( \mathcal{A} \subset \mathbb{N}^A \) is a chain, if it is linearly ordered by the relation \( \succeq \).

An unbounded system, which is also a chain, is called an unbounded chain. The existence of an unbounded chain follows from Zorn's lemma.

Definition:
Let \( \tau_1 \) be the smallest power of an unbounded chain. Let \( \tau_2 \) be the smallest power of a hereditary unbounded system.

It is clear that \( \tau_1 \leq \tau_2 \leq \tau_3 \leq 2^{\mathbb{N}} \).

Theorem 2.1:
Let \((Q, \mu)\) be an \( \mathcal{L} \)-product of \( \mathcal{L} \) two-point spaces. The following properties are equivalent:
(1) \((Q, \mu)\) is a non- \( F \)-space.
(2) \( \tau_2 \geq \tau_2 \).

Let \((Q, \mu)\) be a space, \( A \subset Q \). We put \( \mu^\alpha A = A \), and for ordinal number \( \alpha \), \( \mu^\alpha A = \mu (\bigcup \mu^\beta A) \).

Theorem 2.2:
Let an \( \mathcal{L} \)-product \((Q, \mu)\) of two-point spaces be a non- \( F \)-space and not countably compact. Then there exists \( A \subset Q \) such that \( \mu^\alpha A \neq A \) for all countable ordinal numbers \( \alpha \).

Definition:
We call a space \((Q, \mu)\) countably regular at a point \( x \).
point $x$, if $x$ is an $R$-point of every subspace $P$ of $(Q, \mathcal{U})$ such that $P = T \cup A \cup \{x\}$, $x \notin V$, $A$ is countable.

We call a space $(Q, \mathcal{U})$ countably regular, if it is countably regular at each of its points.

Theorem 2.3:

Let $(Q, \mathcal{U})$ be an $\aleph$-product of $\aleph$ two-point spaces, $\aleph \geq \aleph_1$. Then $(Q, \mathcal{U})$ and $(Q, \mathcal{U}^*)$ are not countably regular.

Theorem 2.4:

Let $(Q, \mathcal{U})$ be an $\aleph$-product of $\aleph$ two-point spaces. The following properties are equivalent:

1. For every $x \in Q$ there exists a closed set $T \subset Q$ such that $x \notin T$, cc. $c.d. T = \aleph_1$ and if $U$ is a neighborhood of $T$ in $(Q, \mathcal{U})$, then $x \in U$. $U$.

2. $(Q, \mathcal{U})$ is not regular.

3. $(Q, \mathcal{U}^*)$ is not regular.

4. $\aleph \geq \aleph_1$.

Problem: I do not know if $\aleph_1$ is the smallest cardinal number, satisfying the Theorem 2.3.

III. $F$-reduction.

Definition:

Let $(Q, \mathcal{U})$ be a space, $\aleph_1$ a cardinal number. We denote by a symbol $(Q, \mathcal{U})$ every collection $\{x_{\lambda, m} : \lambda \in \Lambda, m \in \mathbb{N}\}$ of elements of $Q$ such that

1. $\cup \mathcal{U} = Q$.

2. There exists a point $x \in Q$ such that $x_{\lambda, m} \rightarrow x$ for every $\lambda \in \Lambda$.

3. If $\{m_i\} \in \mathbb{N}^\mathbb{N}$, $\{\lambda_i\} \in \Lambda^\mathbb{N}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, then $x_{\lambda_i, m_i} \rightarrow x$.

$x$ is an $R$-point of a space $(Q, \mathcal{U})$, if for every neighborhood $U$ of $x$ there exists its neighborhood $V$ such that $U \subset V$.

xx) cf [6], Theorem 1.1.
Theorem 3,1:
Let \((\mathcal{U}, \mu)\) be an \(\mathcal{L}\)-product of \(n\) two-point spaces. Then \((\mathcal{U}, \mu)\) is a discrete space if and only if there exists some \((\mathcal{U}_0, \mu_0)\).

Theorem 3,2:
Let \((\mathcal{U}, \mu)\) be an \(\mathcal{L}\)-product of \(n\) two-point spaces. Let \((\mathcal{V}, \nu) = \bigotimes_{\alpha \in \Lambda} (\mathcal{B}_\alpha, \nu_\alpha)\), and \(\Lambda = \mathbb{N}\), and \(\mathcal{U}_\alpha \subseteq \mathcal{U}\) and every \(\mathcal{U}_\alpha\) contain at least two points. If \((\mathcal{U}, \mu)\) is a discrete space, \((\mathcal{V}, \nu)\) is also a discrete space.

Theorem 3,3:
Let \((\mathcal{U}, \mu)\) be an \(\mathcal{L}\)-product of \(n\) two-point spaces. The following properties are equivalent:

1. \((\mathcal{U}, \mu)\) is discrete.
2. \(\mathcal{U}_\alpha = \mathbb{R}\).

Theorem 3,4:
Let \((\mathcal{V}, \nu) = \bigotimes_{\alpha \in \Lambda} (\mathcal{B}_\alpha, \nu_\alpha)\), and \(\Lambda = \mathbb{N}\), and \(\mathcal{U}_\alpha \subseteq \mathbb{R}\), and every \(\mathcal{U}_\alpha\) contain at least two points. Then \((\mathcal{V}, \nu)\) is a discrete space.

IV. The space of continuous functions.

Now we consider an \(\mathcal{L}\)-product \((\mathcal{U}, \mu)\) and its subspaces, where \((\mathcal{U}, \mu) = \bigotimes_{\alpha \in \Lambda} (\mathcal{B}_\alpha, \mu_\alpha)\) and all \((\mathcal{B}_\alpha, \mu_\alpha)\) are the spaces of real numbers \(\mathbb{R}\) (with a usual topology). We suppose that \(\mathcal{U}\) is also a topological space and consider the space of real continuous functions on \(\mathcal{U}\).

In the following theorems \(C(\mathcal{U})\) denotes the set of all real continuous functions on \(\mathcal{U}\), or the set of all real continuous and bounded functions on \(\mathcal{U}\), or the set of all mappings of \(\mathcal{U}\) into \(\mathbb{R}\); \(D(\mathcal{U})\) denotes any system of real functions on \(\mathcal{U}\). \(\mu\) denotes a topology on \(C(\mathcal{U})\) (event. \(D(\mathcal{U})\)) such that \((C(\mathcal{U}), \mu)\) (event. \(D(\mathcal{U}), \mu\)) is a subspace of a given \(\mathcal{L}\)-product.

\(x)\) cf [6, Theorem 2,2.](/content/7)
Theorem 4.1:
Let \( m \) be a cardinal number. Let \( \forall x \in \mathbb{R} \), for every \( f \in C(P) \). Then \( \forall x \in \mathbb{R} \), for every continuous mapping \( f \) of \( P \) into any separable metric space.

This Theorem implies easily:

Theorem 4.2:
Let a set \( f(P) \) be countable for every \( f \in C(P) \). Then \( (C(P), \mathcal{U}) \) is an \( \mathcal{F} \) -space.

Proposition II,3 in [6] implies easily the following Theorem:

Theorem 4.3:
Let \( D(P) \) satisfy the following conditions:
1) If \( g \in C(E_\mathbb{R}), f \in D(P) \), then \( g \circ f \in D(P) \) (\( g \circ f \) denotes the composition of \( f \) and \( g \)).
2) There exists a function \( f \in D(P) \) such that \( f(P) \) contains a closed subset which is dense-in-itself and non-meager. Then \( (D(P), \mathcal{U}) \) is a non-\( \mathcal{F} \) -space.

This Theorem implies easily:

Theorem 4.4:
Let \( P \) be a compact space, containing an infinite discrete normally imbedded \( \alpha \) subset. Then \( (C(P), \mathcal{U}) \) is a non-\( \mathcal{F} \) -space.

V. \( \mathcal{F} \) -reduction of a space of continuous functions.

Definition:
Let \( D(P) \) be a system of real functions on \( P \), \( k(P) \) the system of all real functions on \( P \), let \( \mu \) be a cardinal number. The symbol \( \langle \mu \rangle (D(P)) \) denotes every collection \( \{ f_x, \mu; x \in X, \mu \in \mathbb{N} \} \) of elements of \( D(P) \) such that
1) \( \mu = \mu \).
2) \( \forall x, y, \mu \in \mathbb{N} \), \( x \neq y \implies \exists \xi \in \mathbb{N} \).

x) A set \( \mathcal{K} \) is said to be normally imbedded in a space \( P \), if \( P \subseteq \mathcal{K} \), and every bounded continuous function on \( \mathcal{K} \) can be extended continuously to \( P \). By discrete subset we mean simply a subset which, as a subspace, contains isolated points only.
3) if \( \{ a_n \} \in \mathbb{N}^\infty \), \( \epsilon_n \in (C(P))^N \), \( f_n \in \Xi \), \( \xi_n \neq \xi \),
for \( i \neq j \), then \( a_i \cdot \epsilon_n \cdot \xi_n \rightarrow A \).

Theorem 5.1:

Let \( (C(P), C(P)) \) be discrete, and \( C(P) = P \).
Then there exists \( \hat{C}_\mu (C(P)) \). Let \( P \) contain a dense subset \( X \), and let \( X = A \), and let \( \hat{C}_\mu (C(P)) \) exist there. Then \( (C(P), C(P)) \) is discrete.

Theorem 5.2:

Let \( A \in D(P) \). Let \( D(P) \) satisfy
a) \( f \in D(P) \Rightarrow \frac{\partial^2 f}{\partial x^2} \in D(P) \)

b) if \( g \in \hat{C}(E_P) \), \( g \) has all derivations, \( g(0) = 0 \), \( g(A) = 1 \), and \( f \in D(P) \), then \( g \circ f \in D(P) \).

Then the following propositions are equivalent:
1) there exists \( \hat{C}_\mu (D(P)) \);
2) there exists \( \hat{C}_\mu (D(P)) \).

Theorems 5.1 and 5.2 imply easily the Theorem II, 1 in [6].

Theorem 5.3:

Let \( P \) be a space, containing a dense countable metrisable subset. Let every neighborhood of every point \( x \in P \) contain a neighborhood of \( x \), which is a dense-in-itself non-meager normal space.

Let \( D(P) \subset C(P) \) such that:
1) \( \mu D(P) = C(P) \) (i.e.: for every \( f \in C(P) \) there exist \( f_n \in D(P) \), \( \mu = 1, 2, \ldots \), such that \( f_n \rightarrow f \)).
2) if \( A \subset P \) is closed, \( g \notin A \), then there is a function \( f \in D(P) \) with \( f(g) = 0 \), \( f(A) = 1 \) for all \( x \in A \).
3) if \( f, g \in D(P) \), then \( f \cdot g \in D(P) \).
4) there exists a function \( \hat{f} \in C(P) \) such that \( \mu \hat{f} = C(P) \rightarrow (f) \).

If more

4) there exists a function \( \hat{f} \in C(P) \) such that \( \hat{f} \neq 0 \), and \( \hat{f} \cdot f \in D(P) \) for all \( f \in D(P) \),

\( \mu \hat{f} \) of \( H_f \) we certainly consider in the space \( (C(P), \mu) \) only. Some non-continuous functions are the limits of sequences of points of \( H_f \), too.
then there exists $H_\Upsilon \subset D(P)$ such that 
\[ \mu \cdot H_\Upsilon = C(P) - (1). \]

This Theorem may be applied, for example, for the set of all real functions of real variables, having all derivations.

VI. Some results about the space $(C(P), \mu)$.

Theorem 6,1:
If a normal space $P$ contains a locally finite disjoint system, the power of which is $\aleph_1$ (event. $\aleph_1$, event. $\aleph_1, \aleph_1$), then $(C(P), \mu)$ and $(C(P), \mu^*)$ are not regular (event. $(C(P), \mu)$ and $(C(P), \mu^*)$) are not countably regular, event. $(C(P), \mu)$ contains a set $H$ such that $\mu \cdot H = \omega + 1$ for all countable ordinal numbers $\alpha$.

Theorem 6,2:
Let $\omega$ be the smallest ordinal number, the power of which is a regular cardinal number $\aleph_1$. Let every subspace of some space $(\alpha, \mu)$ contain a dense subset, the power of which is $< \aleph_1$.
Then there exists an ordinal number $\alpha$ for every $\alpha \in \alpha$ such that $\alpha < \omega$ and $\omega \cdot \alpha \cdot R = \mu^{\omega \cdot \alpha \cdot R}$.

Theorem 6,3:
Let $P$ be a union of the countable number of compact metric spaces. Then every subspace of $(C(P), \mu)$ contains a dense countable subset.

Theorem 6,4 which is a strengthening of Theorem 1,2 in [2], follows immediately from the Theorems 6,2 and 6,3:

Theorem 6,4:
Let $P$ be a union of the countable number of compact metric spaces.
Then, for every $H \subset C(P)$, $\mu \cdot H = \omega + 1$ for some countable $\alpha \cdot H$ and consisting from open sets.$^V$

VII. Countable compactness of $(C(P), \mu)$.
$C(P)$ denotes the set of all continuous mappings of $P$ into $< 0, 1 >$ in this section.

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Theorem 7.1:
Let a space $P$ contain a normally imbedded discrete set of the power $c$. Then $(C(P), \mu)$ is not countably compact.

Theorem 7.2:
Let a normal space $P$ contain a closed $F_\sigma$ subset which is not open. Then $(C(P), \mu)$ is not countably compact.

Theorem 7.3:
Let $C = \{\mu\}$.

a) If $P$ is a perfectly normal space, then $(C(P), \mu)$ is countably compact only for a countable discrete $P$.

b) If $P$ is a normal space and $(C(P), \mu)$ is a non-$F_\sigma$ space, then $(C(P), \mu)$ is not countably compact.

VIII. Borel functions.

Let $P$ be a perfectly normal space. $B(P)$ denotes the set of all real Borel functions on $P$, or the set of all real bounded Borel functions (or bounded by a certain constant), or the set of all characteristic functions of Borel subsets of $P$. A definition of the topology $\mu$ on $B(P)$ is evident.

Theorem 8.1:
Let us suppose that a perfectly normal space $P$ contains a normally imbedded discrete subset, the power of which is $c = 2^\aleph_0$, let card. $\nu \leq 2^\nu$. Then $(B(P), \mu)$ is a discrete space.

Theorem 8.2:
If a perfectly normal space $P$ contains a Borel subset, which may be mapped continuously on a topological product of $n$ two-point spaces, card. $\nu \leq 2^n$, then $(B(P), \mu)$ is a discrete space.

It is clear that the Theorems 6.1; 6.3; 6.4; 7.1 hold also for $(B(E), \mu)$.

The problem, raised by J. Novák, whether $(B(E), \mu)$ is regular, remains unsolved. It may be shown only, that there...
exists a subspace \( P \) of \( E_1 \) such that \( (B(P), \omega) \), and 
\( (v(P), \omega^+) \) are not regular (neither are they countably regular).

References.

[4] J. Novák, L. Mišík: O \(\mathcal{L}\) -priestorach spojitých funkc-