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ON THE LOGARITHMIC POTENTIAL

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Introductory remark. Suppose we are given a simple oriented curve C of finite length and a continuous function F on it. It is well known that the corresponding potential of the double distribution can be defined by the formula

$$W_C(z; F) = \mathcal{Im} \int_C \frac{F(\xi)}{\xi - z} \, d\xi , \quad z \in E_2 - C .$$

Investigation of the behaviour of $W_C(z; F)$ (as well as of the corresponding Cauchy's integral) as z approaches C is of importance for a number of applications. Accordingly, solution of the following problems I - III seems to be of interest.

Problem I. Fix a point $\xi \in C$ and suppose that

$$C \cap \{z ; z = \xi \pm \rho \exp i\vartheta, \ 0 < \rho < R\} = \emptyset$$

for every

$$\vartheta \in (\vartheta_0 - \delta, \vartheta_0 + \delta) = \{\vartheta ; \vartheta_0 - \delta < \vartheta < \vartheta_0 + \delta\} \ (\delta > 0).$$

Put $\mathcal{U} = \{z ; z = \xi + \rho \exp i\vartheta_0 , \ 0 < \rho < R\}$.

Find a necessary and sufficient condition to secure the existence of

$$\lim_{\substack{z \to \xi \\ z \in \mathcal{U}}} W_C(z; F)$$

for every continuous function F on C .

Problem II. Suppose that C is a simple closed curve and write G for its bounded complementary domain. What (necessary and sufficient) restrictions are to be imposed on C that $W_C(z; F)$ be uniformly continuous on G for every continuous distribution F on C ? Or, which is the same: Under what conditions $W_C(z; F)$ (so far considered for $z \in G$ only) can be extended to a continuous function on $\overline{G} = G \cup C$ whenever F is continuous on C ?

The problem II having been settled, one can consider the operator

$$TF(\xi) = \sigma F(\xi) - \lim_{\substack{z \to \xi \\ z \in G}} W_C(z; F)$$

on the Banach space B of all continuous functions F on C equipped with the norm $\|F\| = \max_{\xi \in C} |F(\xi)|$; here $\sigma = \pi \ (-\pi)$ provided C is positively (negatively) oriented respectively. In connection with the classical Fredholm's method for solution of the Dirichlet problem it is useful to have an expression (or, at least, some estimates) for the quantity

$$(1) \qquad \omega = \inf_{L} \|T - L\|,$$

L ranging over the system of all completely continuous linear operators acting on B . (ω^{-1} is so-called Fredholm's radius of T .) Thus we arrive at the following

P r o b l e m III. Find an expression for ω making clear its dependence of the shape of C .

It is the purpose of the present paper to show that methods of Real Variables make it possible to solve the problems quoted above. The problem I will be treated in a slightly more general fashion for path-curves with any number of self-intersections. We shall introduce certain geometric indicatrices and variations (which can also be described purely analytically) and announce some theorems solving I - III in terms of them.

D e f i n i t i o n 1. Let φ be a continuous complex-valued function on $\langle a, b \rangle = \{t;\ t \in E_1,\ a \leqq t \leqq b\}$ and let f be a real-valued function on $\langle a, b \rangle$. Given a point $z \in E_2 - \varphi(\langle a, b \rangle)$ we fix a continuous real-valued function ϑ_z on $\langle a, b \rangle$ with

$$(2)\quad |\varphi(t) - z| \exp i\,\vartheta_z(t) = \varphi(t) - z,\ a \leqq t \leqq b,$$

and define

$$(3)\qquad w_\varphi(z; f) = \int_a^b f(t)\, d\,\vartheta_z(t)$$

provided the Stieltjes integral on the right-hand side e - xists.

R e m a r k . The definition (3) is independent of the choice of ϑ_z fulfilling (2).

D e f i n i t i o n 2. Let φ have the same meaning as in the definition 1. Fix $\xi \in E_2$ and define on $\langle 0, 2\pi \rangle$ the function $\mu^\varphi(\alpha; \xi)$ of the variable α as follows: For $\alpha \in \langle 0, 2\pi \rangle$ put

$\mu^\varphi(\alpha; \xi) = n$ (where $n \geqq 0$ is an integer) if and only if the path-curve φ meets the half-line $\{z;\ z = \xi + \rho \exp i\alpha,\ \rho > 0\} = P_\xi^\alpha$ exactly n-times. Further put $\mu^\varphi(\alpha; \xi) = +\infty$ if φ meets P_ξ^α infinitely many times. Thus $\mu^\varphi(\alpha; \xi)\ (0 \leqq \mu^\varphi(\alpha; \xi) \leqq +\infty)$ is equal to the number of points in $\varphi^{-1}(P_\xi^\alpha)$ Similarly, define for any $r > 0$ the function $\mu_r^{\varphi}(\alpha; \xi)$ of the variable α on $\langle 0, 2\pi \rangle$ as follows: For every $\alpha \in \langle 0, 2\pi \rangle$, $\mu_r^{\varphi}(\alpha; \xi)$ $(0 \leqq \mu_r^{\varphi}(\alpha; \xi) \leqq +\infty)$ is equal to the number of points in

$$\varphi^{-1}(P_{\xi}^{\alpha} \cap \{z;\ |z - \xi| < r\}).$$

R e m a r k . It can be proved that (for fixed φ and ξ) the functions $\mu^{\varphi}(\alpha; \xi)$, $\mu_r^{\varphi}(\alpha; \xi)$ $(r > 0)$ are measurable. Therefore we may introduce the following

N o t a t i o n .

$$v^{\varphi}(\xi) = \int_0^{2\pi} \mu^{\varphi}(\alpha; \xi)\, d\alpha,$$

$$v_r^{\varphi}(\xi) = \int_0^{2\pi} \mu_r^{\varphi}(\alpha; \xi)\, d\alpha,$$

the integrals on the right-hand side being taken in the sense of Lebesgue.

D e f i n i t i o n 3. The meaning of φ, ξ is the same as in the definition 2. For any $\rho > 0$ denote by $\nu^{\varphi}(\rho; \xi)$ $(0 \leqq \nu^{\varphi}(\rho; \xi) \leqq +\infty)$ the number of points in $\{t;\ t \in \langle a, b \rangle,\ |\varphi(t) - \xi| = \rho\}$. The function $\nu^{\varphi}(\rho; \xi)$ (φ, ξ are fixed) is measurable on $(0, +\infty)$ and, consequently, the Lebesgue integrals

$$\int_0^{\infty} \nu^{\varphi}(\rho; \xi)\, d\rho = u^{\varphi}(\xi),$$

$$\int_0^{r} \nu^{\varphi}(\rho; \xi)\, d\rho = u_r^{\varphi}(\xi) \quad (r > 0)$$

are available.

Now we are able to announce the following

T h e o r e m 1. Let φ be a continuous complex-valued function on $\langle a, b\rangle$, $\xi \in \varphi(\langle a, b\rangle)$. Put $U = \{z;\ z = \xi + \rho \exp i\vartheta_0,\ 0 < \rho < R\}$. Suppose that the set $\varphi^{-1}(\xi)$ is finite and that there exists a $\delta > 0$ such that $\varphi(\langle a,b\rangle) \cap \{z;\ z = \xi \pm \rho \exp i\vartheta,\ 0<\rho<R\} = \emptyset$ whenever $|\vartheta - \vartheta_0| < \delta$. If

$$\lim_{\substack{z\to\xi\\ z\in U}} w_\varphi(z; f)$$

exists for every continuous (real-valued) function f on $\langle a, b\rangle$, then

$$v^\varphi(\xi) < +\infty, \tag{4}$$

$$\sup_{r>0} r^{-1} u_r^\varphi(\xi) < +\infty. \tag{5}$$

The converse of this theorem is also true. More precisely, we have the following

T h e o r e m 2. Let φ be a continuous complex-valued function on $\langle a, b\rangle$, $\xi \in \varphi(\langle a, b\rangle)$ and suppose that the set $\varphi^{-1}(\xi) = \{t_1 < \ldots < t_n\}$ is finite. If (4) holds, then there exist the limits

$$\lim_{t\to t_k+} \frac{\varphi(t)-\varphi(t_k)}{|\varphi(t)-\varphi(t_k)|} = \tau^\varphi(t_k+) \quad \text{whenever } t_k < b,$$

$$\lim_{t\to t_k-} \frac{\varphi(t)-\varphi(t_k)}{|\varphi(t)-\varphi(t_k)|} = \tau^\varphi(t_k-) \quad \text{whenever } t_k > a.$$

For the sake of simplicity, let us agree to write

$\tau^\varphi(t_1-) = \tau^\varphi(a+)$ in the case $t_1 = a$, $\tau^\varphi(t_n+) = \tau^\varphi(b-)$ in the case $t_n = b$. If, moreover, the condition (5) takes place, then, for every continuous function f on $\langle a, b\rangle$, $w_\varphi(\xi + r\xi;\ f)$ tends to a limit as

$r \to 0+$ uniformly with respect to ξ on any compact set

$$K \subset \{\xi\,;\,\xi \in E_2,\ |\xi| = 1\} - \bigcup_{k=1}^{n} \{\tau^{\varphi}(t_k-), \tau^{\varphi}(t_k+)\}.$$

R e m a r k . If $\varphi^{-1}(\xi) \subset (a, b)$, in the above theorem, $\lim_{r \to 0+} w_{\varphi}(\xi + r\xi\,;\, f)$ is constant on every component of $\{\xi\,;\,|\xi| = 1\} - \bigcup_{k=1}^{n} \{\tau^{\varphi}(t_k-), \tau^{\varphi}(t_k+)\}$. The same is true provided $\varphi(a) = \varphi(b)$

R e m a r k . If φ is a rectifiable path-curve on $\langle a, b \rangle$, then the set of all $\xi \in \varphi(\langle a, b \rangle)$ with $\sup_{r > 0} r^{-1} u_r^{\varphi}(\xi) = +\infty$ is of (Hausdorff) linear measure zero. On the other hand, example can be given of a simple rectifiable path-curve φ on $\langle a, b \rangle$ such that the set $\{\xi\,;\,\xi \in \varphi(\langle a, b \rangle),\ v^{\varphi}(\xi) = +\infty\}$ is of positive linear measure. This, of course, does not mean, that there exists a continuous function F on $C = \varphi(\langle a, b \rangle)$ such that non-tangential limits of $W_C(z; F)$ do not exist on a set of positive linear measure. (In fact, the contrary is known to be true.)

By theorems 1, 2 the problem I is solved. Let us now proceed to the problem II.

N o t a t i o n . From now on we shall assume that φ is a complex-valued function on $\langle a, b \rangle$ such that $\varphi(a) = \varphi(b)$ and $\varphi(t_1) \neq \varphi(t_2)$ whenever

$$0 < |t_1 - t_2| < b - a,\ t_1, t_2 \in \langle a, b \rangle .$$

The symbol C will be used to denote the set $\varphi(\langle a, b \rangle)$ as well as the oriented curve determined by φ . For any continuous real-valued function F on C and any $z \in E_2 - C$

we put $W_C(z; F) = w_\varphi(z; f)$, where

$$f(t) = F(\varphi(t)), a \leqq t \leqq b.$$

This is, clearly, in accordance with the notation used in the above remarks. Further we shall write G for the bounded complementary domain of C and B for the Banach space of all continuous real-valued functions F on C with the usual norm $\|F\| = \max_{\xi \in C} |F(\xi)|$.

T h e o r e m 3. Suppose that $W_C(z; F)$ is uniformly continuous on G whenever F is continuous on C . Then

$$\sup_{\xi \in C} v^\varphi(\xi) < +\infty \tag{6}$$

and (since $v_r^\varphi(\xi) \leqq v^\varphi(\xi)$) $\sup_{\xi \in C} v_r^\varphi(\xi) < +\infty$ for every $r > 0$.

Conversely, the following theorem holds.

T h e o r e m 4 . Suppose that

$$\sup_{\xi \in C} v_r^\varphi(\xi) < +\infty$$

for a certain $r > 0$. Then (6) holds and, for every continuous function F on C , $W_C(z; F)$ $(z \in G)$ can be extended to a continuous function on $G \cup C$. The operator T on B defined by

$$TF(\xi) = \sigma F(\xi) - \lim_{\substack{z \to \xi \\ z \in G}} W_C(z; F), \quad \xi \in C$$

($\sigma = \pm \pi$ according to whether C is positively or negatively oriented) is bounded and its norm is equal to $\sup_{\xi \in C} v^\varphi(\xi)$.

As to the problem III, the following theorem can be proved.

T h e o r e m 5 . Suppose that (6) holds and let ω have the same meaning as in (1). Then

$$\omega = \lim_{r \to 0+} \sup_{\xi \in C} v_r^{\varphi}(\xi) .$$

Proofs of the above theorems together with further results in this direction will appear later.