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REMARK ON TOPOLOGICAL EMBEDDING OF COMMUTATIVE MAPPINGS

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Throughout this remark X will denote a topological space and F the system of all continuous mappings from X into X ; F will be considered as a subset of the product X^X and will be endowed with the "pointwise topology", i.e. the relativised product topology. Accordingly, the pointwise convergence of nets will be considered ($f_\alpha \to f$ means that $f_\alpha(x) \to f(x)$ for every $x \in X$).

If $G \subset F$, $Y \subset X$, then $G(Y)$ denotes the set of all $g(y)$, $g \in G$, $y \in Y$. If $x \in X$, we shall write $G(x)$ instead of $G(\{x\})$. The set $G(x)$ will be called the orbit of x under G .

An orbit cover of Y under G is defined to be a class $\mathfrak{a}$ of subsets of Y such that

(1) $Y = \cup \mathfrak{a}$,

(2) every set from $\mathfrak{a}$ is an orbit of some $y \in Y$ under G .-The operation in all semigroups will be the composition of mappings.-The cardinal of a system $\mathfrak{a}$ is denoted by card $\mathfrak{a}$.

We shall prove the following theorems:

Theorem 1. Let G, $G \subset F$, be a commutative semigroup, and X be an orbit of e, $e \in X$, under G . Then G with the pointwise topology is homeomorphic to X .

Theorem 2. Let G, $G \subset F$, be a commutative semigroup and $\mathfrak{a}$ be an orbit cover of X under G . Then G with the pointwise topology is homeomorphic to a subset of $X^{\operatorname{card} \mathfrak{a}}$

(with the product topology), provided, G contains identity.

<u>Proof of theorem 1</u>. For $g \in G$, put $\varphi(g) = g(e)$; clearly, φ maps G onto X.

If $\varphi(g_1) = \varphi(g_2)$ for some $g_1, g_2 \in G$, then $g_1(e) = g_2(e)$. For every $x \in X$ we can find $g \in G$ such that $g(e) = x$. Hence

$g_1(x) = g_1[g(e)] = g[g_1(e)] = g[g_2(e)] = g_2[g(e)] = g_2(x)$,

and $g_1 = g_2$. Therefore φ is one-to-one.

We are going to prove that φ is a homeomorphism.

Let $\{f_\alpha, \alpha \in D\}$ be a net, $f, f_\alpha \in G$ for $\alpha \in D$. If $f_\alpha(e) \to f(e)$, then $f_\alpha(x) \to f(x)$ for every $x \in X$. Clearly, for every $x \in X$ we can find $g \in G$ such that $g(e) = x$. We have

$$f_\alpha(x) = f_\alpha[g(e)] = g[f_\alpha(e)]$$

and $f_\alpha(x) \to f(x)$, as g is assumed continuous. Therefore φ is open. If $f_\alpha(x) \to f(x)$ for every $x \in X$, $f, f_\alpha \in G$, then $\varphi(f_\alpha) \to \varphi(f)$, and the theorem is proved.

<u>Proof of theorem 2</u>. Let $Y \in \mathcal{A}$, $G(y) = Y$. Evidently $G[G(y)] = Y$. We shall denote by $G|Y$ the class of all mappings from G restricted to Y. $G|Y$ is a commutative semigroup of continuous mappings from Y into Y, $G|Y(y) = Y$. According to the preceding theorem there exists a homeomorphism φ_Y from $G|Y$ onto Y. Let us define the mapping φ from G into $X^{\text{card } \mathcal{A}}$ coordinatewise:

$$\varphi_Y(g) = \varphi_Y(g|Y) \quad \text{for every } Y \in \mathcal{A}.$$

If $g_1, g_2 \in G$, $g_1 \neq g_2$, then there exists $Y \in \mathcal{A}$ such that $g_1|Y \neq g_2|Y$, hence $\varphi_Y(g_1) \neq \varphi_Y(g_2)$, as φ_Y is one-to-one.

Therefore φ is a one-to-one mapping from G onto $\varphi(G)$. It is sufficient to prove that φ is both continuous and open.

Let $f_\alpha \to f$, $f, f_\alpha \in G$. Then $\varphi_Y(f_\alpha) \to \varphi_Y(f)$ for every $Y \in \mathfrak{a}$. Let $\varphi(f_\alpha) \to \varphi(f)$, $f, f_\alpha \in G$. To every $x \in X$ there exists $Y \in \mathfrak{a}$, $G(y) = Y$, such that $x \in Y$. We have $\varphi_Y(f_\alpha) \to \varphi_Y(f)$, and $f_\alpha(y) \to f(y)$. We can write $x = g(y)$, $g \in G$. Then $f_\alpha(x) = f_\alpha[g(y)] = g[f_\alpha(y)]$, and $f_\alpha(x) \to f(x)$, as g is continuous. The proof is concluded.