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ON SOME SPECTRAL PROPERTIES OF RADON-NICOLSKI OPERATORS AND THEIR GENERALIZATIONS

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1. Introduction

1.1. Let X be a complex Banach space. We shall denote its elements by small Roman characters; the zero element will be denoted 0. The set of linear bounded operators mapping space X into itself also forms a Banach space, which we shall denote $[X]$ (similarly as in [6]). The norm in $[X]$ is defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|, \quad T \in [X].$$

Unless the contrary is stated we shall use the denotations and definitions introduced in [6] .

1.2. Definition [3]. The value $\mu_0 \in \sigma(T)$ is called a dominant point of the spectrum $\sigma(T)$ of the operator T, if the inequality $|\lambda| < |\mu_0|$ holds for every point $\lambda \in \sigma(T), \quad \lambda \neq \mu_0$.

2. Radon-Nicolski operators

2.1. Definition. Operator $T \in [X]$ is called Radon-Nicolski operator, if it can be expressed as $T = \mathcal{U} + V$, where $V \in [X]$; $\mathcal{U}$ is a linear compact operator mapping X into itself and $r_\sigma(T) > r_\sigma(V)$ for the spectral radii $r_\sigma(T), r_\sigma(V)$.

2.2. The Radon-Nicolski operators have some of the spectral properties of compact operators. We shall list such of these which will be of use below. Let $T = U + V$ be a Radon-Nicolski operator and T' the operator adjoint to T .

Then we have

(a) Every point λ , for which $|\lambda| > r_\sigma(V)$ is either a regular point of the operator T or an isolated pole of the resolvent $R(\lambda, T)$ and the corresponding projector $E[\lambda, T]$ has a finite-dimensional range $\mathcal{R}(E[\lambda, T])$.

(b) Operator T' is Radon-Nicolski operator.

(c) The eigenvalue λ_0, $|\lambda_0| > r_\sigma(V)$ of the operator is a simple eigenvalue if and only if the equations

(1) $\lambda_0 x - Tx = 0, \quad \lambda_0 y' - T'y' = 0$

have no orthogonal solutions, i.e. $x \neq 0$, $y' \neq 0$: $y'(x) \neq 0$.

The proof of (a) can be found in [1], lemma 2, page 709; (b) is evident and (c) can be proved similarly as in the case of compact operators. (See also [4], [5].)

It is easy to prove, that (a) and (c) also hold for operators, some iteration of which is a Radon-Nicolski operator. The authors of [7] call such operators strongly quasi-compact.

Let T be a closed linear, generally unbounded operator mapping the domain $\mathcal{D}(T) \subset X$ into X . We shall investigate the spectral properties of operator T' under the assumption that $f(T)$ is a Radon-Nicolski operator for some function $f \in \mathfrak{A}_\infty(T)$.

2.3. Let $f \in \mathfrak{A}_\infty(T)$, let $f(T) = \mathcal{U} + V$ be a Radon-Nicolski operator and $\lambda_0 \in \sigma(T)$, $|f(\lambda_0)| > r_\sigma(V)$. Then λ_0 is a pole of the resolvent $R(\lambda, T)$ and the projector B_1 corresponding to the spectral set $\{\lambda_0\}$ has a finite-dimensional range i.e. the dimension of the eigenmanifold of the operator T , corresponding to the value λ_0 is finite.

Proof. Let $S = f(T)$, $\mu_0 = f(\lambda_0)$. According to theorem [6] 5.71 – A page 302 we have $\mu_0 \in \sigma(S)$. Let $\sigma = \sigma_e(T) \cap \{f^{-1}(\mu_0)\}$, where $\sigma_e(T)$ is the extended spectrum of the operator T. According to (a) of § 2.2 μ_0 is an isolated point of the spectrum $\sigma(S)$, $\{\mu_0\}$ is the spectral set of operator S and according to theorem [6] 5.71 – D page 304 the corresponding projector is

$$E_\sigma = \frac{1}{2\pi i}\int_C R(\lambda, T)\,d\lambda,$$

where C is the boundary of the Cauchy domain D, containing the set σ and not having any other common points with the spectrum $\sigma(T)$. Let $X_\sigma = \mathcal{R}(E_\sigma)$ be the range of the operator E_σ and S_0 the restriction of operator S onto X_σ. According to theorem [6] 5.7 – B page 299 $\sigma(S_0) = \{\mu_0\}$, so that $\kappa_\sigma(S_0) = |\mu_0| > \kappa_\sigma(V)$. For the restriction V_0 of operator V we have $\kappa_\sigma(V_0) \leqq \kappa_\sigma(V)$ and thus S_0 is also Radon-Nicolski operator. Thus if $|\lambda|$ is large enough, we have for all λ except perhaps a countable number of isolated points the expression

$$R(\lambda, S_0) = [I - R(\lambda, V_0)\,\mathcal{U}_0]^{-1} R(\lambda, V_0).$$

Let C_0 be a circle with its centre in λ_0, with such a radius that C_0 lies in the region $|\lambda| > \kappa_\sigma(V)$ and no point of the spectrum $\sigma(T)$ except λ_0 lies inside C_0 or on C_0. The operator-function $[I - R(\lambda, V_0)\,\mathcal{U}_0]^{-1}$ exists and is analytical in every point $\lambda \in C_0$. For λ with $|\lambda|$ large enough identity

$$[I - R(\lambda, V_0)\,\mathcal{U}_0]^{-1} = I + R(\lambda, V_0)\,\mathcal{U}_0\,[I - R(\lambda, V_0)\,\mathcal{U}_0]^{-1}$$

holds. With the help of an analytical continuation we can extend this to C_0. Thus

$$R(\lambda, S_0) = R(\lambda, V_0) + R(\lambda, V_0)\,\mathcal{U}_0\,[I - R(\lambda, V_0)\,\mathcal{U}_0]^{-1} R(\lambda, V_0).$$

Since $R(\lambda, V_o)$ is analytical on C_o, we obtain

$$E[\lambda_o, S_o] = \frac{1}{2\pi i} \int_{C_o} R(\lambda, S_o)\, d\lambda =$$

$$= \frac{1}{2\pi i} \int_{C_o} R(\lambda, V_o)\, \mathcal{U}_o [I - R(\lambda, V_o)\, \mathcal{U}_o]^{-1} R(\lambda, V_o)\, d\lambda \, .$$

Since according to the assumption $\mathcal{U}_o$ is a compact operator, the integrand is also compact for all $\lambda \in C_o$. As a uniform limit of compact operators - the Riemann sums - $E[\lambda_o, T]$ is a compact operator. Space X_σ must have a finite dimension since $E[\lambda_o, T]$ is an identity - operator in X_σ.

Let T_1 be the restriction of operator T onto X_σ. According to theorem [6] 5.7 - B page 299, $\sigma_e(T_1) = \sigma$. The set $\sigma_e(T)_1$ evidently cannot be the whole extended plane, since $\sigma_e(T_1) \subset \sigma_e(T)$ but the resolvent set $\rho(T)$ is not empty. It follows from here that $\mathcal{D}(T) \supset X_\sigma$, for otherwise the range of the operator $\lambda I - T_1$ would not form the whole X_σ for any λ and this is not possible. Thus $T_1 \in [X_\sigma]$ and $\sigma = \sigma(T_1)$ is a finite set. Since $\lambda_o \in \sigma$, λ_o is an isolated point of the spectrum of operator T. According to corollary VII.3.21 of [1] $E_\sigma = B_1 + P$, $B_1 P = \Theta$, $P B_1 = \Theta$, where Θ denotes the zero-operator, P is the projector corresponding to the set $\sigma - \{\lambda_o\}$ and B_1 is defined in the formulation of the theorem. Since $\mathcal{R}(B_1) \subset X_\sigma$, $\mathcal{R}(B_1)$ has a finite dimension. The restriction of operator T onto $\mathcal{R}(B_1)$ has the property, that λ_o is a pole of its resolvent and hence that in a certain basis this restriction is determined by a square matrix with a finite number of rows. It is not difficult to obtain from here that λ_o is a pole of the resolvent $R(\lambda, T)$.

Since the eigenmanifold $\mathcal{N}(\lambda_0 I - T)$ is a part of $\mathcal{R}(B_1)$ it must also have a finite dimension.

The proved theorem is an analogy of theorem [6] 5.8 - F page 312, where it is supposed that T is a compact operator for a suitable function $f \in \mathcal{O}_{\infty}(T)$.

2.4. We have already mentioned strongly quasicompact operators the spectral properties of which are similar to those of a Radon-Nicolski operators. For a given strongly quasicompact operator $T \in [X]$, $T^m = \mathcal{U} + V$, $m \geq 1$, it is easy to find a function $f \in \mathcal{O}_{\infty}(T)$ such that $f(\lambda) = \lambda^m$ in the neighborhood of the spectrum $\sigma(T)$ i.e. $f(T) = T^m$, so that the property (a) of strongly quasicompact operators mentioned in § 2.2. is a consequence of the theorem proved in § 2.3 . Another important example is the class of operators with the property, that their resolvents for some λ are Radon-Nicolski operators. Let T be a linear, in general unbounded operator, the resolvent $R(\lambda, T)$ of which is a Radon-Nicolski operator for λ in some region $\Gamma \subset \rho(T)$. Let $\alpha \in \Gamma$ and $R(\alpha, T) = \mathcal{U} + V$ be a Radon-Nicolski operator. Then $R(\alpha, T) = f(T)$, where $f(\lambda) = (\alpha - \lambda)^{-1}$. According to the theorem of § 2.3, every point $\lambda_0 \in \sigma(T)$, for which $|\alpha - \lambda_0| > \kappa_\sigma(V)$ is a pole of the resolvent $R(\lambda, T)$ and the corresponding projector $E[\lambda_0, T]$ has a finite-dimensional range. Examples of such operators can be found in physical applications, for instance in some boundary problems.

2.5. In this paragraph we shall investigate the case: $T \in [X]$ and there exists such an $f \in \mathcal{O}_{\infty}(T)$ that $f(T) = \mathcal{U} + V$ is Radon-Nicolski operator. According to theorem (a) of § 2.3 every point λ_0 of the spectrum $\sigma(T)$ for which $|f(\lambda_0)| > r_\sigma(V)$ is a pole of the resolvent and the corresponding projector $E[\lambda_0, T]$ has a finite dimensional range. It is also easy to prove the modification of assertion (c) of 2.3 .

The eigenvalue λ_0, $|f(\lambda_0)| > r_\sigma(V)$ of operator T is simple if and only if equations (1) have no orthogonal solutions. This assertion is used in the proof of the existence of positive eigenvectors of operators reproducing a cone in a Banach space.

3. K -positive operators

In this chapter we use the definitions of [2].

3.1. Let Y be a real Banach space and K a cone in space Y . Let X be a complex extension of space Y , i.e. the space of pairs $z = x + iy$, $x \in Y$, $y \in Y$ $(i^2 = -1)$ with a norm defined as

$$\|z\| = \sup_{0 \le \vartheta \le 2\pi} \|x \cos\vartheta + y \sin\vartheta\|$$

or with an equivalent norm.

If T is a linear operator mapping space Y into itself, we define its complex extensions (denoted by the same symbol) by the formula

$$Tz = Tx + iTy, \quad z = x + iy .$$

Evidently $T \in [X]$, if T is a bounded linear operator mapping Y into itself.

Further let $K \subset Y$ be a "productive" cone in space Y. Operator $T \in [Y]$ is called K-positive, for short positive, if $Tx \in K$ for $x \in K$.

3.2. Let $T \in [Y]$, $TK \subset K$ and $f \in \mathfrak{A}_\infty(T)$ be such a function, that $f(T) = U + V$ is a Radon-Nikolski operator.

Then a positive eigenvalue μ_0 lies in the spectrum of operator T and

$$|\lambda| \leqq \mu_0, \quad \lambda \in \sigma(T).$$

At least one eigenvector $x_0 \in K$, $\|x_0\| = 1$ of the operator T corresponds to the eigenvalue μ_0 and at least one eigenfunctional $y_0' \in K'$ $\|y_0'\| = 1$ (K' is the cone adjoint to cone K) of the operator T':

$$Tx_0 = \mu_0 x_0, \quad T' y_0' = \mu_0 y_0'.$$

Proof. We shall prove that operator T has at least one eigenvalue. We have assumed that $f(T) = \mathcal{U} + V$ is a Radon-Nicolski operator. Hence an isolated point $\lambda_0 \in \sigma(f(T))$, $|\lambda_0| > r_\sigma(V)$ exists. Let $\sigma = \sigma(T) \cap \{f^{-1}(\lambda_0)\}$. According to theorem [6] 5.71 - D, the projectors $E_\sigma = E[\sigma, T]$ and $E[\lambda_0, f(T)]$ are identical. It follows that a point $\mu_0 \in \sigma(T)$ exists such that $f(\mu_0) = \lambda_0$. According to the theorem of § 2.3, μ_0 is a pole of the resolvent $R(\lambda, T)$ and thus an eigenvalue of the operator T. Further the proof can be performed similarly as the proof of theorem 6.1 in [2].

3.3. According to e.g. [2] the cone K - is volume type if it has interior points. The operator $T \in [Y]$ is called strongly K-positive, for shor strongly positive, if for every vector $x \in K$, $x \neq 0$ a natural number $p = p(x)$ exists, such that vector $T^p x$ is an interior element of the cone K.

Space Y can be partially ordered with the help of the cone K. We define that $y \succ x$ if $y - x \in K$. If K is a volume-type cone and $y - x$ is an interior element of K, we write $y \succ\succ x$. Evidently $y \succ x$ follows from $y \succ\succ x$.

3.4. Let us assume that the following conditions are satisfied:

1) K is a volume-type cone.

2) Operator $T \in [Y]$ is strongly positive.

3) Such a $f \in \mathcal{O}l_{\infty}(T)$ exists that $f(T) = \mathcal{U} + V$ is Radon-Nicolski operator.

Then: (α) Operator T has just one eigenvector x_0 ; $\| x_0 \| = 1$, inside K .

(β) The adjoint operator T' has just one eigenfunctional y_0' , $\| y_0' \| = 1$; $T' y_0' = \mu_0 y_0'$, $y_0'(x) > 0$ for $x \in K$, $x \neq 0$.

(γ) The eigenvalue μ_0 corresponding to the eigenvectors x_0 , y_0' is simple and

$$|\lambda| < \mu_0$$

for all $\lambda \in \sigma(T), \lambda \neq \mu_0$ (μ_0 is a dominant point of the spectrum of operator T).

On the other hand, if T satisfies condition 3 and has the properties (α), (β), (γ), then T is strongly positive.

Theorem 3.4 is the same as theorem 6.3 in [2], only the assumption of [2] that T is a compact operator is re - placed by the weaker assumption 3 .

Theorem 3.4 can be proved similarly as theorem 6.3 in [2]. It is only necessary to ensure the existence of an ei - genvalue μ_0 of the operator T fulfilling the condition $|f(\mu_0)| > \kappa_\sigma(V)$. According to the theorem of § 3.2 , operator T - has an eigenvalue $\mu_0 > 0$ for which

$$|\lambda| \leq \mu_0$$

if $\lambda \in \sigma(T)$. An eigenvector $x_0 \in K$ of the operator T

and an eigenfunctional $y_0' \in K'$ of operator T' corresponds to the eigenvalue μ_0. The rest of the proof is the same as the corresponding part of the proof of theorem 6.3 in [2].

3.5. Let us demonstrate a further interesting and important in applications spectral property of strongly positive operators.

Let $T \in [Y]$ and

$$(2)\quad R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_0)^k T_k + \sum_{k=1}^{\infty} (\lambda - \mu_0)^{-k} B_k$$

be the Laurent series for the resolvent $R(\lambda, T)$ in the neighborhood of an isolated singularity μ_0. It is well-known ([6], p. 305) that $T_k \in [X]$ for $k = 0, 1, \dots$ and

$$B_1 = \frac{1}{2\pi i} \int_{C_1} R(\lambda, T)\, d\lambda, \quad B_{k+1} = (T - \mu_0 I) B_k, \ k = 1, 2, \dots,$$

where C_1 is the boundary of the circle K_1 with the property $\overline{K_1} \cap \sigma(T) = \{\mu_0\}$ (symbol $\overline{K_1}$ means the closure of set K_1).

If $f \in \mathfrak{A}_\infty(T)$, $f(T) = \mathcal{U} + V$ is a Radon-Nicolski operator and μ_0 is the dominant eigenvalue of the strongly positive operator T, then $B_k = \theta$ for $k \geq 2$, where θ is a zero operator.

Let the following conditions be satisfied:

1) K is a volume-type cone in Y.
2) $T \in [Y]$ is a strongly positive operator.
3) Function $f \in \mathfrak{A}_\infty(T)$ is such, that $f(T) = \mathcal{U} + V$ is a Radon-Nicolski operator.
4) μ_0 is the dominant eigenvalue of operator T.

Then operator B_1 in expression (2) for the resolvent is strongly positive.

Proof. We shall prove that for $x \in K$, $x \neq 0$ we have $B_1 x \succ\succ 0$. According to lemma 6.1 of [2] a positive constant c, independent on m, exists such that

$$\| \mu_0^{-m} T^m x \| = \| \mu_0^{-(n-p)} T^{(n-p)} y \| \geq c > 0 ,$$

where $y = T^p x$. It follows from assumption 2 that $y \succ\succ 0$ for a suitable non-negative p. According to theorem 1 in [3] the norm of the vector $\mu_0^{-m} T^m x$ converges to zero:

$$\| \mu_0^{-m} T^m x - B_1 x \| \leq \| \mu_0^{-m} T^m - B_1 \| \, \| x \| \to 0 .$$

Thus $B_1 x \neq 0$ and hence $B_1 x = x_0$ is an eigenvector of the operator T corresponding to the value μ_0. According to the theorem of § 3.4 vector x_0 is strongly positive, i.e. $x_0 \succ\succ 0$.

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