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NEUTRON TRANSPORT THEORY IN THE MULTIGROUP ENERGETICAL APPROXIMATION

(Preliminary communication)

Ivo MAREK , Praha

1. Introduction and formulation of the problem

Our purpose in this paper is to establish the system of neutron transport equations in the multigroup energetical approximation. We shall study the case of fast reactors i.e. the case, when all energetical groups of neutrons participate in the fission passing through. We shall use definitions and denotations of the monography [3]. As to the mathematical formulation of the problem and to some assumptions we will use the papers of Vladimirov [6], [7], in which the mathematical theory of one-group approximation of neutron transport equation is given.

The system of multigroup-approximation equations for neutron transport can be symbolically written as

$$L\varkappa = B\varkappa + \lambda C\varkappa + s \,, \tag{1.1}$$

where $\varkappa = (\varkappa_1, \dots, \varkappa_m)$ is the solution-vector and the operators L, B, C will be defined below.

Let G be a bounded open and connected set in the Euclidean space R_n $(n \geq 1)$; let $\dot{G}$ be the boundary of G . By the symbol Ω will be denoted the set of normed vectors $\vec{\Omega} = (\Omega_1, \dots, \Omega_n), |\vec{\Omega}| = 1$. Let us put $\mathcal{U} = G \times \Omega$. We define the operators L_j , $j = 1, \dots, m$ on the set $\mathcal{U}$ as follows:

$$L_j \varkappa_j \equiv \vec{\Omega} \nabla \varkappa_j + \Sigma_j \varkappa_j \,, \quad \Sigma_j \geq c > 0$$

with the boundary conditions

(1.2) $$\varkappa_j(\vec{r},\vec{\Omega}) = 0 \quad \text{for} \quad \vec{r} \in \dot{G}\,, \quad (\vec{\Omega},\vec{n}) < 0\,,$$

where the symbol $\vec{n}$ denotes the direction of the exterior normal to the boundary $\dot{G}$ and $(\vec{\Omega},\vec{n})$ denotes the cosinus of the angle between the vectors $\vec{\Omega}, \vec{n}$.

Remark. The condition (1.2) can be written in the given form only if some further conditions with regard to the boundary $\dot{G}$ are satisfied. In the following paragraphs we shall formally use (1.2) in the general case, but we mean by this the exact formulation of the boundary conditions given in [6].

We define the operators

(1.3) $$B_{jk}\varkappa_k = \int_\Omega \Sigma_k^s w_{kj}^s(\vec{\Omega},\vec{\Omega}')\varkappa_k \, d\vec{\Omega}'\,,$$

(1.4) $$C_{jk}\varkappa_k = \int_\Omega \Sigma_k^f \nu_k^f w_{kj}^f(\vec{\Omega},\vec{\Omega}')\varkappa_k \, d\vec{\Omega}'$$

and we put

(1.5) $$\nu_{kj}^s = \Sigma_k^s w_{kj}^s\,, \quad \nu_{kj}^f = \Sigma_{kj}^f \nu_k^f w_{kj}^f\,.$$

We shall suppose that the kernels (1.5) have the properties (a) - (d) of the paper [7] p. 683.

The operators L, B, C will be defined as operator-matrices

(1.6) $$L = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_m \end{pmatrix}, \quad B = (B_{jk}), \quad C = (C_{jk})\,.$$

Besides the assertions on the existence of the solutions of the homogeneous ($s = \mathfrak{A}$) and the inhomogeneous ($s \neq \mathfrak{A}$) systems (1.1, we shall give similar assertions for the adjoint system

(1.7) $$L^*x^* = B^*x^* + \lambda C^*x^* + s^*\,,$$

where L^*, B^*, C^* are the adjoint operators to the L, B, C

in the Hilbert space $X = L_2(\Omega) \times \dots \times L_2(\Omega)$ with the scalar product $(x,y) = \sum_{j=1}^{m} (x_j, y_j)_j$, where $(x_j, y_j)_j$ is the scalar product in the space $L_2(\Omega)$ for the j-th coordinate of vectors $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$. We shall give general iteration methods for the construction of the mentioned solutions of the systems (1.1) and (1.7).

2. Auxiliary lemmas

The symbol $[X]$ denotes the space of linear bounded operators mapping X into itself. Let $K \subset X$ be the cone [2] of non negative vector-functions on Ω. The spectrum of the linear operator T will be denoted by $\sigma(T)$.

To an arbitrary vector $x = (x_1, \dots, x_m)$ we shall define a vector $|x|$ with the coordinates $(|x_1|, \dots, |x_m|)$, where $|x_j|$ is the absolute value of the function x_j, so that

$$|x_j(\vec{r}, \vec{\Omega})| = x_j(\vec{r}, \vec{\Omega}) \cdot \exp\{-i \arg x_j(\vec{r}, \vec{\Omega})\}.$$

If $x \in X$, $y \in X$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, the inequality

$$x < y$$

means that

$$x_j(\vec{r}, \vec{\Omega}) \leqq y_j(\vec{r}, \vec{\Omega})$$

for almost all $(\vec{r}, \vec{\Omega}) \in \Omega$ and the sharp inequality holds at least for one index j_0 in a set having positive measure.

Linear bounded operator $T \in [X]$ is called K-positive, if the vector $Tx = y \in K$ for any $x \in K$. K-positive operator T is called absolutely K-positive, if T has the following properties:

(α) For any vector $x \in K$, $x \neq \theta$ and for any $\varepsilon > 0$ there exists a natural N so that the measure of the set of

the zero-points of the vector-function $T^N x$ is not greater than ε .

(β) For any x , for which $\arg x_j \neq const$ at least for one index j_0 the inequality

(2.1) $$|Tx| < T|x|$$

holds.

Lemma 2.1 (fundamental lemma). An absolutely K-positive compact operator T with the positive spectral radius $r(T)$ has a positive simple eigenvalue μ_0 and an eigenvector x_0 positive almost everywhere in Ω corresponds to this value. An almost everywhere positive in Ω eigenvector x_0^* of the adjoint operator T^* corresponds to the same value μ_0. The following relations

$$Tx_0 = \mu_0 x_0 \ , \ T^* x_0^* = \mu_0 x_0^* \ ,$$

(2.2) $$|\lambda| < \mu_0 \ , \ |\lambda^*| < \mu_0$$

are correct for $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$, $\lambda^* \in \sigma(T^*)$, $\lambda^* \neq \mu_0$. If the eigenvectors y_0, y_0^* of the operators T, T^* lie in K , then $y_0 = c_0 x_0$, $y_0^* = c_0^* x_0^*$, where c_0 , c_0^* are positive constants.

The application of the fundamental lemma to the systems (1.1) and (1.7) consists in using of the mentioned lemma for the operators

(2.3) $$T = (L-B)^{-1} C \ , \ T^* = C^* (L^* - B^*)^{-1}$$

(see chapter 3).

Let Γ be an interval of real numbers. The operator-function $T = T(\gamma), \gamma \in \Gamma, T(\gamma) \in [X]$ is called continuous in the point $\gamma_0 \in \Gamma$, if for any $\varepsilon > 0$ such a $\sigma > 0$ exists

so that

$$\| T(\gamma) - T(\gamma_0) \| < \varepsilon$$

for $|\gamma - \gamma_0| < \sigma$. If $T = T(\gamma)$ is continuous in every point $\gamma \in \Gamma$, we shall call T continuous in Γ .

Lemma 2.2. Let us assume that

1. For any $\gamma \in \Gamma$, $T(\gamma) \in [X]$ is an absolutely K - positive compact operator.

2. Operator-function $T = T(\gamma)$ is continuous in Γ .

3. A vector $u_1 \in K$ exists such that $u - u_1 \in K$, where u $(\| u \| = 1)$ is the eigenvector of the operator $T(\gamma)$ corresponding to the dominant eigenvalue $\mu_0(\gamma)$ and the relation

$$\{ [T(\gamma') - T(\gamma)] u_1 - \nu(\gamma, \gamma') u_1 \} \in K$$

holds for $\gamma \in \Gamma, \gamma' \in \Gamma, \gamma < \gamma'$ where $\nu(\gamma, \gamma') > 0$.

Then a positive dominant eigenvalue-function $\mu_0 = \mu_0(\gamma)$ of the operator-function $T = T(\gamma)$ is continuous and purely monotonous in Γ i.e. the inequality

$$\mu_0(\gamma) < \mu_0(\gamma')$$

holds for $\gamma < \gamma'$.

The value $\gamma_0 \in \Gamma$ is called critical parameter of the compact absolutely K -positive operator-function $T = T(\gamma)$ if the identity $\mu_0(\gamma_0) = 1$ holds for the corresponding dominant eigenvalue $\mu_0(\gamma_0)$ of the operator $T(\gamma_0)$.

From the lemma 2.2 follows the existence or the non existence of the critical parameters. If they do exist, their unicity follows from the same lemma.

3. Existence of the solutions of homogeneous systems

In this chapter we shall give some assertions from which the validity of the fundamental lemma 2.1 for the operators

(2.3) follows. Thus we obtain the fundamental result - the existence of a positive dominant simple eigenvalue of the systems (1.1), (1.7).

Lemma 3.1. The operator $L^{-1}C$ is absolutely K-positive, compact and its spectral radius $r(L^{-1}C)$ is positive.

From this lemma immediately follows

Theorem 3.1. Let be $r(L^{-1}B) < 1$. Then a positive dominant simple eigenvalue $\mu_0 = \lambda_0^{-1}$ of the homogeneous systems (1.1) (1.7) ($\theta = \theta$, $\theta^* = \theta$) exists. An eigenvector $x_0 \in K$ of the system (1.1) and an eigenvector $x_0^* \in K$ of the system (1.7) correspond to this eigenvalue. Besides the mentioned eigenvectors the homogeneous systems (1.1), (1.7) do not have other eigenvectors in K .

Remark. It is easy to show that the assumption $r(L^{-1}B) < 1$ in theorem 3.1 is natural and that the relation $r(L^{-1}B) \geq 1$ is physically impossible.

4. Existence of the solutions of the inhomogeneous systems

From the results of the third chapter it follows that all λ , for which

(4.1) $$|\lambda| < \mu_0^{-1}$$

are regular points of the systems (1.1) and (1.7). We can obtain the solutions of the mentioned systems in the following form

(4.2) $$x = R(1, \lambda T)(L - B)^{-1} s ,$$

(4.3) $$C^* x^* = R(1, \bar{\lambda} T^*) C^* (L^* - B^*)^{-1} s^*$$

where $R(\lambda, T) = (\lambda I - T)$, $R(\lambda, T^*) = (\lambda I - T^*)$ and s, s^* are arbitrary vectors in X .

Theorem 4.1. For λ , for which (4.1) holds, the system (1.1) or (1.7) has one and only one solution x or x^* for arbitrary $s \in X$ or $s^* \in X$.

For arbitrary λ the system (1.1) or (1.7) has a solution x or x^* then and only then, if

$$(s, y^*) = 0 \quad \text{or} \quad (s^*, y) = 0$$

for any solution y^* or y of the homogeneous system

$$L^* y^* = B^* y + \bar{\lambda} C^* y \quad \text{or} \quad L y = B y + \lambda C y .$$

5. Iteration methods

Among the numerical methods for solving the systems (1.1) and (1.7) the iteration methods are very important. In the case of homogeneous systems (1.1), (1.7) one of the often used iteration methods is the so called method of source-iteration. This method is a certain mathematical model of the physical processes described by systems (1.1) (1.7). Mathematically the source-iteration method is a modification of Kellogg's method for the construction of eigenvalues and eigenvectors of linear operators. The iteration formulae presented here contain or generalize most of the used iteration processes. The convergence of the mentioned processes can be derived from a single general principle (see [4]).

Generally we can define the source iteration method as follows:

$$(5.1) \qquad L x^{(n)} = B x^{(n)} + C x^{(n-1)}, \quad x^{(0)} \in K, \quad x^{(0)} \neq \vartheta ,$$

$$(5.2) \qquad L^* x^{*(n)} = B^* x^{*(n)} + C^* x^{*(n-1)}, \quad x^{*(0)} \in K, \quad x^{*(0)} \neq \vartheta ,$$

$$(5.3) \qquad x_{(n)} = \frac{x^{(n)}}{(x^{(n)}, x^{(0)})} , \quad x^*_{(n)} = \frac{x^{*(n)}}{(x^{*(n)}, C x^{(0)})} ,$$

(5,4) $$\lambda_{(n)} = \mu_{(n)}^{-1} = \frac{(Cx^{(n)}, y'_n)}{(Cx^{(n+1)}, x'_n)},$$

where y'_n, x'_n form sequences of vectors in X, which converge weakly to the same vector $y' \in X$:

(5,5) $$(x, y'_n) \to (x, y'), \; (x, x'_n) \to (x, y') \text{ for } x \in X.$$

Let be

(5,6) $$(x^{(0)}, B_1^* y) \neq 0,$$

where $$B_1^* = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T^*)\, d\lambda ;$$

C_0 denotes a circle with its center in μ_0 and with the radius ρ_0 such that for $H = \{\lambda \,|\, |\lambda| \leqq \rho_0\}$ the relation

$$H \cap \sigma(T^*) = \{\mu_0\}$$

holds.

Theorem 5.1 The source iteration method defined by formulae (5.1) - (5.4) converges i.e. the following relations hold in the norm of the space X

$$x_{(n)} \to x_0 , \quad x^*_{(n)} \to x_0^*$$
$$\lambda_{(n)} \to \lambda_0 = \mu_0^{-1} ,$$

where x_0, x_0^* are positive solutions of the homogeneous systems (1.1) and (1.7) corresponding to the positive dominant eigenvalue μ_0.

If (4.1) holds, the following iteration processes [5], [6] can be used for the construction of the solutions of the inhomogeneous systems (1.1) and (1.7) besides the usual iterations

(5.7) $$x_{n+1} = \frac{1}{1 - \lambda \mu_{(n)}} \lambda^{n+1} x^{(n+1)} + \sum_{k=1}^{n} \lambda^k x^{(k)} + (L-B)^{-1} s,$$

(5.8) $$x^*_{n+1} = \frac{1}{1-\bar{\lambda}\mu_{(n)}} \bar{\lambda}^{n+1} x^{*(n+1)} + \sum_{k=1}^{n} \bar{\lambda}^k x^{*(k)} + (L^*-B^*)^{-1} s^*,$$

where $x^{(k)}$, $x^{*(k)}$, $\mu_{(n)}$ are defined by formulae (5.1), (5.2), (5.4) and $x^{(0)} = (L-B)^{-1}s$, $x^{*(0)} = (L^*-B^*)^{-1}s^*$.

Theorem 5.2. If the assumption (4.1) holds, the sequence (5.7) or (5.8) converges to the single solution x or x^* of the inhomogeneous system (1.1) or (1.7) i.e.

$$x_{n+1} \longrightarrow x \quad \text{or} \quad x^*_{n+1} \longrightarrow x^*$$

in the norm of the space X .

6. Critical parameters

In the case of nuclear reactors the operators L , B , C , L^*, B^*, C^* in systems (1.1), (1.7) depend on some parameters. We shall illustrate this on an example.

Let us suppose that the operators mentioned above depend on a parameter $\gamma \in \Gamma$, where Γ is an interval of real numbers.

The nuclear reactor is called critical, overcritical or subcritical, depending on whether the corresponding dominant eigenvalue $\mu_0(\gamma)$ of the operator $[L(\gamma)-B(\gamma)]^{-1} C(\gamma)$ is equal, greater or smaller than 1 . It is evident that only a single critical parameter can exist. Formally this follows from the fact that $\mu_0 = \mu_0(\gamma)$ is a purely monotonous function of the argument $\gamma \in \Gamma$.

We shall give an example of the dependence of the criticality of the nuclear reactor on the enriching of the fuel by a more fissionable isotope.

Let us consider the system (1.1) and the following system

(6.1) $$L'x' = B'x' + \lambda' C'x' ,$$

where

$$\Sigma_k = \Sigma_k(\gamma),\ \Sigma_k^f = \Sigma_k^f(\gamma),\ \nu_k^f = \nu_k^f(\gamma),$$

(6.2) $$\Sigma_k' = \Sigma_k(\gamma'),\ \Sigma_k'^f = \Sigma_k^f(\gamma'),\ \nu_k'^f = \nu_k^f(\gamma'),$$

$$\Sigma_k' > \Sigma_k,\ \Sigma_k'^f > \Sigma_k^f,\ \nu_k'^f > \nu_k^f .$$

Theorem 6.1. If the assumptions(6.2) are fulfilled the inequality

$$\lambda_0' < \lambda_0$$

holds for the values $\lambda_0 = \lambda_0(\gamma)$, $\lambda_0' = \lambda_0(\gamma')$.

If an enriching $\gamma' \in \Gamma$ exists such that the corresponding $\lambda_0' = \lambda_0(\gamma') < 1$, then one and only one value of enriching $\gamma_0 \in \Gamma$ exists, for which $\lambda_0(\gamma_0) = 1$.

In conclusion we call the readers' attention to the fact that many similar problems having the form (1.1), where L, B, C are linear operators such that $T = (L-B)^{-1}$ is absolutely K-positive compact operator, can be considered by our method.

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