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ON THE SOLUTION OF HOMOGENEOUS FUNCTIONAL EQUATIONS IN HILBERT SPACE

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This paper contains the proofs of theorems (theorems 1 and 4) which were published previously without proofs in Commentationes Mathematicae Universitatis Carolinae 1,3 (1960) [1].

Let the equation

\[ Ay - \alpha y = \theta \]

be given, where \( A \) throughout this paper will denote a linear operator bounded in complex Hilbert space \( H \), \( \alpha \) is a real parameter. Suppose that \( A \) is a positive operator \((Ay, y) > 0 \) for every \( y \in H \), \( y \neq 0 \) and \((Ay, y)=0 \iff y=\theta \).

This assumption will be later omitted. We solve the equation (1) by iterative process

\[ y_{n+1} = \frac{1}{\alpha_{n+1}} Ay_n \]

where the parameters \( \{\alpha_n\} \), \( n=1,2,\ldots \) are to be determined from the condition that the functional \( \| Ay - \tau y \|^2 \) for the given element \( y=y_0 \in H \) shall catch the minimal value on the set \( \mathcal{R} \) of all real numbers. Let us denote that value of \( \tau \) (dependent on \( n \)) by \( \alpha_n \). We get

\[ \alpha_{n+1} = \frac{(Ay_n, y_n)}{\| Ay_n \|^2} \]

Then

\[ y_{n+1} = \frac{\| y_n \|^2}{(Ay_n, y_n)} Ay_n \quad y_0 \neq 0, \quad y_n \in H, \quad (n=0,1,2,\ldots) \]

Lemma 1. Let \( A \) be a positive operator in \( H \). Then the sequence \( \{\alpha_n\} \) defined by (2), is monotone, increasing...
and convergent.

Proof: The sequence \( \{u_n\} \) is bounded because

\[
u_{n+1} \leq \frac{\|Ay_n\| \|y_n\|}{\|y_n\|^2} \leq \|A\| .
\]

From (2) and (3)

\[
\|Ay_n\| \leq \frac{\|y_n\| \|Ay_n\|}{\|y_n\|^2} \|y_n\| .
\]

Hence \( \|y_n\| \leq \|y_{n+1}\| \) for every \( n \). Since

\[
(Ay_{n-1}, y_{n-1}) \leq \frac{(Ay_{n-1}, y_n)}{\|y_{n-1}\|^2} \|y_n\|^2 \leq (Ay_{n-1}, y_n)
\]

we have from Schwarz's inequality

\[
(Ay_{n-1}, y_{n-1})^2 \leq (Ay_{n-1}, y_n)^2 \leq (Ay_{n-1}, y_{n-1})(Ay_n, y_n)
\]

Thus

\[
(5) \quad (Ay_{n-1}, y_{n-1}) \leq (Ay_n, y_n) \quad \text{for every } n .
\]

From the equality

\[
(Ay_n, y_n) = \frac{(Ay_n, y_n)}{\|y_n\|^2} (y_{n+1}, y_n)
\]

follows that \( \|y_n\|^2 = (y_{n+1}, y_n) \). In view of (2) and of the precedent equality we get

\[
\|y_n\|^2 = \frac{1}{u_n} (Ay_{n-1}, y_n), \quad \|y_n\|^2 = \frac{1}{u_{n+1}} (Ay_n, y_n).
\]

We have now

\[
u_n(Ay_n, y_n) = u_{n+1}(Ay_{n-1}, y_n)
\]

and from (5)

\[
u_n^2(Ay_n, y_n)^2 \leq (u_{n+1}(Ay_{n-1}, y_{n-1})(Ay_n, y_n)) \leq (u_{n+1}(Ay_n, y_n))^2 .
\]

Hence \( u_n \leq u_{n+1} \) for every \( n \). It follows from the fact that \( u_n > 0 \) for every \( n \) and \( A \) is a positive operator. Since \( \{u_n\} \) is increasing and bounded, there exists \( \lim_{n \to \infty} u_n = u \) and \( u_n \leq u \leq \|A\| .

Lemma 2. Let \( A \) be a positive operator in \( H \).

Then the sequence \( \{y_n\} \) defined by (2) is monotone, increasing and bounded.
Proof: Let us denote \( g_n = \frac{y_n}{\|y_n\|} \). According to \( g_{n+1} = \lambda_{n+1} A y_n \), where \( \lambda_{n+1} = \|y_n\|^2/\|y_{n+1}\|(Ay_n, y_n) \). Hence

\[
\frac{\|y_{n+1}\|}{\|y_n\|} = \frac{\|y_n\|}{(y_n, g_{n+1})} = \frac{1}{(g_n, g_{n+1})}.
\]

It is sufficient to show that \( \prod_{n=1}^{\infty} \frac{1}{(g_n, g_{n+1})} \) converges.

Because \( (g_n, g_{n+1}) \leq 1 \), the product converges, when the series \( \sum_{n=1}^{\infty} [1 - (g_{n-1}, g_n)] \) is convergent.

From (6) and (2) we obtain

\[
A y_n = \frac{A y_n}{\|y_n\|} = \frac{\|y_{n+1}\| g_{n+1}}{\|y_n\|} = \frac{1}{(u_{n+1} (g_n, g_{n+1}) g_{n+1}).
\]

From (7), we get

\[
(g_n, g_{n+1}) \frac{(g_n, g_{n+1})}{(u_{n+1})} = \frac{(g_n, g_{n+1})}{(u_{n+1})} (A g_n, y_n) = \frac{(g_n, g_{n+1})}{(u_{n+1})} (g_{n-1}, g_n).
\]

Further

\[
0 \leq (g_n - g_{n-1}, A (g_n - g_{n-1})) = (g_n - g_{n-1}, u_{n+1} \frac{1}{(g_n, g_{n+1})} g_{n+1}) - (g_n - g_{n-1}, u_{n+1} \frac{1}{(g_n, g_{n+1})} g_{n+1}) =
\]

\[
= (u_{n+1} - (u_{n+1}) (g_{n-1}, g_n) - \frac{1}{(g_{n-1}, g_n)} + u_n.
\]

It follows from (8) that

\[
(u_{n+1} + u_n - 2(u_n \frac{1}{(g_{n-1}, g_n)} \geq 0)
\]

and hence

\[
1 - (g_{n+1}, g_n) \leq 1 - \frac{2(u_n)}{(u_n + u_{n+1})} = \frac{u_{n+1} - u_n}{u_n + u_{n+1}} \leq \frac{u_{n+1} - u_n}{2u_n}.
\]

(\( n = 1, 2, \ldots \))
Therefore
\[
\sum_{n=1}^{\infty} [1-(q_{a_{n-1}}, q_{a_n})] \leq \sum_{n=1}^{\infty} \frac{\mu_{n+1} - \mu_n}{2\mu_n} .
\]
The sequence \(\{\mu_n\}\) converges, and hence the series
\[
\sum_{n=1}^{\infty} [1-(q_{a_{n-1}}, q_{a_n})]
\]
is convergent. This concludes the proof.

**Theorem 1.** Let \(A\) be a non-negative ((\(Ay, y\) \geq 0 for every \(y \in H\)) completely continuous operator in a complex Hilbert space \(H\). Let \(N\) be a null set of \(A\) and let \(y_0 \in H \cap N\) be not orthogonal to the eigenspace \(H(\tilde{\mu}_1)\) corresponding to the first eigenvalue \(\tilde{\mu}_1\) of (1).

Then the sequence \(\{\mu_n\}\) defined by (3), (2) is monotone, increasing and it converges to \(\tilde{\mu}_1\). The sequence \(\{y_n\}\) defined by (2), (3), is convergent in \(H \cap N\) to one of the eigenfunctions corresponding to \(\tilde{\mu}_1\).

**Proof:** The inequality \(\|Ay\|^2 \leq \|A\| (Ay, y)\)
holds for every \(y \in H\). Hence \(A\) is a positive operator on \(H \cap N\). According to our assumption \(y_0 \in H \cap N\). Suppose that \(y_{n+1} \in H \cap N\). Then \(\langle A y_{n+1}, y_{n+1} \rangle > 0\) and from (2)
\[Ay_{n+1} = \frac{1}{\mu_{n+1}} A^2 y_n .\]
The null set \(N\) of \(A\) coincides with the null set of \(A^2\). Hence \(y_{n+1} \in H \cap N\).

Now we use lemma 1 and 2. There exists a positive number \(C\) so that \(\|y_n\| \leq C\). The sequence is bounded, because \(\|y_n\| \leq \frac{C}{\mu_n}\). Hence it contains the subsequence \(\{\frac{y_n}{\mu_{n_k}}\}\) such that \(\frac{1}{\mu_{n_k}} Ay_{n_k}\) converges. We set \(\lim \frac{1}{\mu_{n_k}} Ay_{n_k} = \tilde{y}\). Because \(\frac{1}{\mu_{n_k}} Ay_{n_k} - y_{n+1} = 0\) for every \(n\) \((n = 0, 1, 2, \ldots)\), then \(\frac{1}{\mu_{n+1}} Ay_{n+1} - y_{n+1} \to 0\).
Therefore \( y_n \to \tilde{y} \) and according to lemma 1, 
\[ A\tilde{y} = (\mu \tilde{y}) \quad (\tilde{y} \neq 0) \]
We shall prove (see 2, Chapt. XV) that \( \mu = (\tilde{\mu}) \).

Let \( P_\lambda (\lambda = 1, 2, \ldots) \) be projectors from \( H \) on eigenspace \( H_{\tilde{\mu}_{\lambda}} \) corresponding to different eigenvalues \( \tilde{\mu}_{\lambda} \) .

We set
\[ \frac{\tilde{g}_\lambda}{\|P_\lambda g_\lambda\|} = (P_\lambda g_\lambda \neq 0), \quad \text{where } g_\lambda = \frac{y_\lambda}{\|y_\lambda\|}. \]

Then \( \tilde{g}_\lambda \in H_{\tilde{\mu}_{\lambda}}, \quad g_\lambda = \sum \frac{\tilde{g}_\lambda}{\|P_\lambda g_\lambda\|} \|P_\lambda g_\lambda\| = \sum a_{\lambda k} \tilde{g}_\lambda, \)
where \( \sum a_{\lambda k}^2 = 1, \quad a_{\lambda k} = \|P_\lambda g_\lambda\|, \quad a_{01} > 0. \)

According to (9)
\[ g_1 = \sum a_{1k} \tilde{g}_k, \quad \text{where } a_{1k} = \frac{\|y_1\|}{\|y_1\|} a_{01}. \]

Generally
\[ g_n = \sum a_{nk} \tilde{g}_k, \quad \text{where } a_{nk} = \frac{(\mu_k \|y_{n-1}\|}{\|y_n\|} a_{n-1 k}; \quad g_n = \frac{y_n}{\|y_n\|}. \]

Suppose now that \( \mu = (\tilde{\mu}_n \; (n > 1)) \). Since \( y_n \to \tilde{y} \),
then \( g_{m_n} \to \tilde{g} \), where \( \tilde{g} = \frac{\tilde{g}}{\|\tilde{g}\|} \), \( g_n = \sum a_{nk} \tilde{g}_k \)
and \( a_{k n} = \lim_{j \to \infty} a_{m_j n} \quad (n = 1, 2, \ldots). \)

Because \( \tilde{g}_k \in H_{\tilde{\mu}_n}, \quad \tilde{g}_k \in H_{\tilde{\mu}_n} \), then \( (\tilde{g}_k, \tilde{g}_k) = 0 \) for \( k \neq n \). Hence \( \tilde{g} = a_n \tilde{g}_n \) \quad and \( |a_n| = 1 \). From \( a_{n k} \geq 0 \)
follows that \( a_n = 1 \) and \( \tilde{g} = \tilde{g}_n \). From (9) we get
\[ (a_n) \]

Further \( \lim_{j \to \infty} a_{m_j n} = a_n = 1, \quad \lim_{j \to \infty} a_{m_j 1} = a_1 = \Theta \).

So that
\[ \lim_{j \to \infty} \frac{a_{m_j n}}{a_{m_j 1}} = \infty \]
This is a contradiction with (10) which shows that \( \mu = \tilde{\mu}_1 \).

Let us denote \( \kappa = \tilde{\mu}_1 - \mu_k \), then
\[
\|g_n - \tilde{g}_n\| = 2(1 - a_{n,1}) \leq 2(1 - a_{n,1}^2) = 2 \sum_{k=2}^{\infty} a_{n,k}^2 \leq \frac{2}{k} \sum_{k} (\tilde{\mu}_1 - \mu_k)^2 \leq \frac{2}{k} (\tilde{\mu}_1 - \mu_{n+1}^2) \rightarrow 0.
\]

Hence \( g_n \rightarrow \tilde{g}_n \), where \( \tilde{g}_n \neq 0 \). By lemma 2 the sequence \( \{y_n\} \) converges and \( \lim_{n \to \infty} \|y_n\| = \sup_{n} \|y_n\| = h > \theta \).

We have \( y_n = g_n \|y_n\| \rightarrow \tilde{g}_n, h \). Hence the sequence \( \{y_n\} \) converges to eigenfunction \( \tilde{v} \) corresponding to \( \tilde{\mu}_1 \). The theorem 1 has been now established.

Let the equation
\[ Ay - \lambda By = \theta \]
be given, where \( A, B \) (not necessarily bounded) are linear operators in \( H \).

**Theorem 2.** Let \( B \) be a linear operator such that \( B^{-1} \) exists and let \( T = B^{-1} A \) be a non-negative completely continuous operator in \( H \). Let \( N \) be a null set of \( T \) and let \( y_0 \in H \otimes N \) be not orthogonal to the eigenspace \( H \tilde{v}_1 \) corresponding to the first eigenvalue \( \tilde{\mu}_1 \) of \( T \). Then the sequence \( \{\mu_n\} \) defined by the equalities
\[
y_{n+1} = \frac{1}{\mu_{n+1}} Ty_n, \quad \mu_{n+1} = \frac{(Ty_n, y_n)}{\|y_n\|^2}
\]
is monotone, increasing and it converges to \( \tilde{\mu}_1 \). The sequence \( \{y_n\} \) converges in \( H \otimes N \) to one of the eigenfunctions corresponding to \( \tilde{\mu}_1 \).

Let \( H \) be a real Hilbert space. We say that an operator \( A \) is symmetrizable by a positive operator \( B \), if the
equality $(B A x, y) = (X, B A y)$ holds for every $x, y \in H$. We define on $H$ a new inner product:

$\langle x, y \rangle = (B x, y)$. 

The product (11) defines on the set of all $x, y \in H$ a new Hilbert space $\mathcal{H}$ which is not generally complete.

Adding to $\mathcal{H}$ the limit points, we get a complete Hilbert space. We denote it by $\mathcal{H}_0$.

The norm in $\mathcal{H}_0$ is defined by the equality

$\|y\|_{\mathcal{H}_0} = (B y, y)^{\frac{1}{2}}$.

**Lemma 3**. ([3],[4]) Let $A$ be a bounded operator in $H$. Then $A$ is bounded in $\mathcal{H}$ and $\|A\|_{\mathcal{H}} \leq \|A\|$.

The operator $A$ is bounded and symmetric in $\mathcal{H}$. It can be extended to the self-adjoint operator $\tilde{A}$ in $\mathcal{H}_0$.

**Lemma 4**. ([3],[4]) The spectrum of the operator $\tilde{A}$ in $\mathcal{H}_0$ is a subset of the spectrum of $A$ in $H$.

**Lemma 5**. ([3],[4]) Let $A$ be a completely continuous operator in $H$. Then $\tilde{A}$ is completely continuous in $\mathcal{H}_0$. The sets of eigenvalues of $A$ in $H$ and $\tilde{A}$ in $\mathcal{H}_0$ are identical. The eigenspaces of $A$ in $H$ and $\tilde{A}$ in $\mathcal{H}_0$ corresponding to the eigenvalue $\mu_r$ are equal.

Hence in view of lemma 5 we may investigate instead the eigenvalues and eigenfunctions of the symmetrizable completely continuous operator $A$ in $H$ the eigenvalues and eigenfunctions of the self-adjoint completely continuous operator $\tilde{A}$ in $\mathcal{H}_0$.

**Theorem 3**. Let $A$ be a completely continuous operator which is symmetrizable by a positive operator $B$ in a real Hilbert space $H$. Let $B A$ be a positive operator in...
Let the equation
\[(12) \quad y - \lambda A y = 0\]
be given, where \(\lambda\) is a parameter, \(A\) a linear bounded operator in \(H\). To solve it, I.A. Birger used the iterative formula
\[(13) \quad y_n = \lambda_n A y_{n-1}, \quad \lambda_n = \frac{(Ay_{n-1}, y_{n-1})}{\|Ay_{n-1}\|^2},\]
where \(\lambda_n\) are Schwarz's parameters. Let \(N\) be a null set of \(A\). We prove the following theorem.

**Theorem 4.** Let \(\tilde{A}\) be a non-negative completely continuous operator in complex Hilbert space \(H\). If an element \(y_0 \in H \cap N\) is not orthogonal to the space \(H_{\tilde{\lambda}_1}\) generated by characteristic functions corresponding to the first characteristic number \(\tilde{\lambda}_1\) of (12), then the sequence \(\{\tilde{\lambda}_n\}\) is monotone, increasing and convergent to \(\tilde{\lambda}_1\).
The sequence \( \{y_n\} \) is convergent in \( \mathcal{H} \Theta \mathcal{N} \) to one of the characteristic functions corresponding to \( k \).

**Proof:** Because
\[
\|y_n\| = \left( \left\langle Ay_{n-1}, y_{n-1} \right\rangle \right)^{1/2} \leq \|y_{n-1}\|,
\]
we have
\[
(14) \quad \|y_n\| \leq \|y_{n-1}\| \leq \ldots \leq \|y_0\|.
\]
The sequence \( \{\|y_n\|\} \) is decreasing and bounded. Therefore it is convergent. Let us denote \( \lim_{n \to \infty} \|y_n\| = \lambda \) . According to (13)
\[
(15) \quad \lambda_n (Ay_{n-1}, y_{n-1}) = \|y_n\|^2, \quad \lambda_{n+1} (Ay_n, y_{n+1}) = \|y_{n+1}\|^2.
\]
Hence
\[
(16) \quad \lambda_{n+1} (Ay_n, y_{n+1}) \leq \lambda_n (Ay_{n-1}, y_{n-1}),
\]
and from (14) we get
\[
\lambda_{n+1} \|Ay_n\|^2 \leq \lambda_n \|Ay_{n-1}\|^2,
\]
so that
\[
(17) \quad \lambda_{n+1} (Ay_n, y_{n+1}) \leq \lambda_n (Ay_{n-1}, y_{n-1}) \leq \ldots \leq \lambda_0 (Ay_0, y_0)
\]
in view of (13). The sequence \( \{\lambda_n (Ay_{n-1}, y_{n-1})\} \) is decreasing and bounded. Hence it converges. From (13) follows that
\[
(18) \quad (Ay_{n-1}, y_n) = (Ay_{n-1}, y_{n-1}) \text{ for every } n = 1, 2, \ldots.
\]
According to (17) and (18)
\[
\lambda_{n+1} (Ay_{n-1}, y_{n-1}) \leq \lambda_n (Ay_{n-1}, y_{n-1}) (Ay_n, y_n) \leq \lambda_{n+1} (Ay_{n-1}, y_{n-1})^2 = \lambda_n (Ay_{n-1}, y_{n-1})^2.
\]
Hence \( \lambda_{n+1} \leq \lambda_n \) for every \( n \) (\( n = 1, 2, \ldots \)). In view of (18) and from the fact that \( \lambda_n > \theta \) and that \( A \) is a positive operator in \( \mathcal{H} \Theta \mathcal{N} \), the sequence \( \{\lambda_n\} \) is decreasing and bounded. There exists \( \lim_{n \to \infty} \lambda_n = \lambda \) and \( \lambda \geq 0 \).

Further according to (13)
From (17) and in view of (15) we have

(19) \[ \| \lambda_n A y_{n-1} - y_n \| = \| y_n \|^2 - \lambda_n (A y_{n-1}, y_{n-1}) . \]

From \( \| y_n \| \to \kappa \) and in view of (17), (15) we have

(20) \[ \| \lambda_n A y_{n-1} - y_n \|^2 \to 0 \quad \text{when} \quad n \to \infty . \]

The sequence \( \{ \lambda_n y_n \} \) is bounded:

\[ \| \lambda_n y_n \| \leq \lambda_n \| y_0 \| = \text{Const.} \]

It contains the subsequence \( \{ \lambda_{n_k} y_{n_k} \} \) such that

\( \{ \lambda_{n_k} A y_{n_k} \} \) converges. Let us denote \( \lim_{k \to \infty} \lambda_{n_k} A y_{n_k} = \tilde{y} \).

From (20) \( y_{n_k} \to \tilde{y} \). Because \( A y_{n_k} \to A \tilde{y} \) and \( \lambda_n \to \lambda \), we get that \( \tilde{y} - \lambda A \tilde{y} = 0 \). We shall prove that \( \lambda > 0 \) and \( \tilde{y} \neq 0 \).

From (18) follows that

(21) \[ 0 < (A y_0, y_0) \leq \ldots \leq (A y_{n-1}, y_{n-1}) \leq (A y_n, y_n) \leq \ldots \]

The sequence \( \{ (A y_n, y_n) \} \) is increasing and bounded:

\[ (A y_n, y_n) \leq \| A \| \| y_n \|^2 \leq \| A \| \| y_0 \|^2 . \]

There exists \( \lim_{n \to \infty} (A y_n, y_n) = \rho \) and \( \rho > 0 \).

According to (13) and (18)

(22) \[ (A y_{n-1}, y_n) = \| A y_{n-1} \| \cdot \| y_n \| \]

for every \( n \) \( (n = 1, 2, \ldots) \). From (22), (18) and (21)

\[ \| A y_{n-1} \| \cdot \| y_n \| \leq (A y_n, y_n) \leq \| A y_n \| \cdot \| y_n \| , \]

so that

\[ 0 < \| A y_0 \| \leq \| A y_1 \| \leq \ldots \leq \| A y_n \| \leq \ldots , \]

\[ \| A y_n \| \leq \| A \| \| y_n \| = \| A \| \| y_0 \| . \]

Hence the sequence \( \{ \| A y_n \| \} \) is increasing and bounded. There exists \( \lim_{n \to \infty} \| A y_n \| = q \) and \( q > 0 \). Since

\[ \lambda_n \to \frac{\rho}{q^2} \quad \text{and} \quad \frac{\rho}{q^2} = \lambda \], then \( \lambda > 0 \).
From the fact that \( \lambda = \inf \frac{\lambda_n}{n} \) and from (21), (15) and (18) we get
\[
\|y_n\|^2 = \lambda_n(Ay_{n-1}, y_{n-1}) \geq \lambda(Ay_0, y_0) > 0.
\]
Since \( y_n \to \tilde{y} \), we have that \( \|y_n\| \to \|\tilde{y}\| \) and \( \|\tilde{y}\|^2 \geq \lambda(Ay_0, y_0) > \theta \). Hence \( \tilde{y} \neq 0 \). Further the proof can be performed similarly as the proof of theorem 1.

H.F. Bückner [6] investigated the iterative process (13) for linear and non-linear problems. I. Marek [7], [8] generalized the methods (3), (13) for bounded operators which have a dominant eigenvalue.

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References


