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ON CYCLIC AND RADIAL VARIATIONS OF A PLANE PATH
(Preliminary communication)
Josef KRÁL, Praha

We shall identify the set of finite complex numbers with the Euclidean plane $E_2$. By a path we mean any continuous complex-valued function $\psi$ on a compact interval $<a, b>$. Given such a path $\psi$, a point $z \in E_2$ and a real number $\alpha$, we denote by $u^\psi(\alpha; z)$ the number (possibly zero or infinite) of points in

$\{t; t \in <a, b>, \psi(t)-z = |\psi(t)-z| \exp i\alpha, \psi(t) \neq z\}$. 

Then $u^\psi(\alpha; z)$ is Lebesgue measurable with respect to the variable $\alpha$ on $<0, 2\pi>$ and we may put

$$v^\psi(z) = \int_0^{2\pi} u^\psi(\alpha; z) \, d\alpha.$$ 

In a similar way, let $v^\psi(\rho; z)$ stand for the number of points in $\{t; t \in <a, b>, |\psi(t)-z| = \rho\} (0 \leq v^\psi(\rho; z) \leq +\infty)$. Since $v^\psi(\rho; z)$ is Lebesgue measurable with respect to $\rho$ on $(0, +\infty)$, we may form the integrals

$$u^\psi_x(z) = \int_0^x v^\psi(\rho; z) \, d\rho; \quad x > 0.$$ 

$v^\psi(z)$ will be called the cyclic variation of $\psi$ with respect to $z$ and $u^\psi(z) = u^\psi_\infty(z)$ will be called the radial variation of $\psi$ with respect to $z$. The quantities $v^\psi(z)$, $u^\psi_x(z)$ were introduced in Comment. Math. Univ. Carolinae 3,1 (1962), 3-10 in connection...
with investigations concerning the boundary behaviour of the logarithmic potential. In present paper we shall collect some results on \( v^\psi(z), \ u^\psi(z) \) which are useful in above mentioned topics in potential theory and may be of some interest in themselves. We shall start with several simple theorems connecting rectifiability of \( \psi \) and the behaviour of \( v^\psi(z), \ u^\psi(z) \).

**Theorem 1.** Suppose that the points \( z^1, z^2, z^3 \) are not situated on a single straight-line. If \( \psi \) is a path and

\[
v^\psi(z^1) + v^\psi(z^2) + v^\psi(z^3) < + \infty
\]

then \( \psi \) is rectifiable.

**Theorem 2.** Let \( z \neq z^2 \) and let \( \psi \) be a path not meeting the straight-line through \( z^1 \) and \( z^3 \). If

\[
v^\psi(z^1) + v^\psi(z^2) < + \infty
\]

then \( \psi \) is rectifiable.

**Theorem 3.** If \( \psi \) is a rectifiable path on \( \langle a, b \rangle \) then \( v^\psi(z) \), considered as a function of the variable \( z \), is finite and locally Lipschitzian on \( E_2 - \psi(\langle a, b \rangle) \).

Further

\[
\lim_{|z| \to \infty} v^\psi(z) = 0.
\]

**Remark 1.** In theorem 3, \( v^\psi \) need not be finite on \( \psi(\langle a, b \rangle) \). If we do not assume that \( \psi \) is rectifiable then \( v^\psi \) may be infinite in some points of \( E_2 - \psi(\langle a, b \rangle) \) and may be discontinous there. This can be shown by simple examples.

Similar theorems hold about the radial variation \( u^\psi \).

**Theorem 4.** Let \( \psi \) be a rectifiable path on \( \langle a, b \rangle \). Then \( u^\psi(z) \) is a finite and
continuously function of the variable $z$ on $E_z$. Moreover, $u^\psi(z)$ is locally Lipschitzian on $E_z - \psi(<a,b>)$.

**Remark 2.** It is easily shown by an example that, in theorem 4, $u^\psi$ need not be Lipschitzian on $E_z$. If the assumption that $\psi$ be rectifiable is dropped then $u^\psi$ may be infinite and discontinuous in some points of $E_z - \psi(<a,b>)$.

**Theorem 5.** Let $z^1 = z^2$ and suppose that $\psi$ is a path not meeting the straight-line through $z^1$ and $z^2$. If

$$\int u^\psi(z^1) + u^\psi(z^2) < +\infty$$

then $\psi$ is rectifiable.

**Theorem 6.** Suppose that $\psi$ is a path and $z^1, z^2, z^3$ are points in $E_z$ not lying on a single straight-line. If

$$\int u^\psi(z^1) + u^\psi(z^2) + u^\psi(z^3) < +\infty$$

then $\psi$ is rectifiable.

**Remark 3.** Simple example can be given of a path of infinite length such that $u^\psi(z)$ is finite for an infinity of points $z$; according to theorem 6, all $z$'s with $u^\psi(z) < +\infty$ must then be situated on the same straight-line.

**Theorem 7.** Let $\psi$ be a path and suppose that

$$\int u^\psi(z) + v^\psi(z) < +\infty$$

for a certain $z \in E_z$. Then $\psi$ is rectifiable.

Now we shall announce two estimates concerning $v^\psi(z)$ and $u^\psi_*(z)$ which are useful for the investigation of non-tangential limits of the logarithmic potential of the double distribution.
Theorem 8. Let \( \psi \) be a path on \(<a, b\>\), \( z \in \mathbb{E}_2 \).

Fix \( \beta \in \mathbb{E}_1 \), \( \kappa > 0 \) and put \( \xi = z + \kappa \exp i \beta \).

Suppose that

\[
(1) \quad z + \rho \exp i \alpha \notin \psi(<a, b>)
\]

whenever \( 0 < \rho < \kappa \), \( |\alpha - \beta| < \sigma \left( 0 < \sigma < \frac{\pi}{2} \right) \). Then

\[
\nu^\psi(z) \leq K [ \nu^\psi(z) + \nu^\psi(\xi) ]
\]

with \( K \) depending on \( \sigma \) only.

Theorem 9. Let \( z \in \mathbb{E}_2 \), \( \kappa > 0 \) and suppose that \( \psi \) is a path not meeting \( \{ \eta; \eta \in \mathbb{E}_1, |\eta - z| > \kappa \} \). Fix a \( \beta \in \mathbb{E}_1 \) and assume that (1) takes place for every couple \( \rho, \alpha \) with \( 0 < \rho < \kappa \), \( |\alpha - \beta| < \sigma \left( 0 < \sigma < \frac{\pi}{2} \right) \). Let \( 0 < \chi \leq \kappa \) and put \( \xi = z + \chi \exp i \beta \).

Then

\[
\nu^\psi(\xi) \leq L \left\{ \nu^\psi(z) + \sup_{0 < t < \chi} \nu^{\psi}(z) \right\}
\]

with \( L \) depending on \( \sigma \) only.

Theorem 10. If \( \psi \) is a path on \(<a, b\>) \text{ then} \nu^\psi(z) \text{ is lower semicontinuous on } \mathbb{E}_2 \text{. If } \psi \text{ is rectifiable then } \nu^\psi(z) \text{ is finite, continuous and subharmonic on } \mathbb{E}_2 - \psi(<a, b>) \text{.}

Notation. If \( \psi \) is a path on \(<a, b\>) \text{ and } \xi \in \mathbb{E}_2 \text{ we shall denote by } N_\psi(\xi) \text{ the number of points in } \{ t; t < a, \xi \}, \psi(t) = \xi \} (0 \leq N_\psi(\xi) \leq +\infty \).

We say that \( \psi \) is simple provided \( \psi(t_1) + \psi(t_2) \) whenever \( a \leq t_1 < t_2 \leq b, \ t_2 - t_1 < b - a \). \( \psi \) is termed closed if \( \psi(a) = \psi(b) \).

Theorem 11. Let \( \psi \) be a path on \(<a, b\>\), \( C = \psi(<a, b>) \) and suppose that \( \sup_{\xi \in C} N_\psi(\xi) < +\infty \).
In order that \( v^\psi(z) \) be bounded on \( E_z - C \) it is necessary and sufficient that it be bounded on \( C \).

As a consequence of theorem 11 we obtain the following Corollary. Let \( \psi \) be a simple closed path on \( <a, b> \) and let \( G \) be a complementary domain of \( C = \psi(<a, b>) \). Then the following conditions \( (a) \) and \( (b) \) are equivalent to each other:

\[
(a) \quad \sup_{z \in C} v^\psi(z) < + \infty ,
\]

\[
(b) \quad \sup_{z \in G} v^\psi(z) < + \infty .
\]

Remark 4. The above corollary makes it possible to prove a necessary and sufficient condition that, for every continuous double distribution on \( C \), the corresponding logarithmic potential be uniformly continuous on \( G \) (cf. theorems 3 and 4 in Comment. Math. Univ. Carolinae 3,1 (1962), p. 9). The above corollary can also be used to obtain a solution of the problem \( N^0 \) 4 raised by I. Babuška in Časopis pro pěstování matematiky 79 (1954), \( N^0 \) 2, p. 164.

The following alternative of theorem 11 is also useful in potential theory.

Theorem 12. Let \( \psi \) be a path on \( <a, b> \), \( C = \psi(<a, b>) \) and let \( G \subset E_z \) be a bounded open set disjoint with \( C \). Write \( B(G) \) for the boundary of \( G \) and suppose that \( N^\psi(\xi) \) is bounded on \( C \cap B(G) \). Further suppose that a neighborhood \( \mathcal{U}(\xi) \) can be associated with every \( \xi \in C \cap B(G) \) that the following condition \( (c) \) be fulfilled:

\[
(c) \quad \text{For every } z \in \mathcal{U}(\xi) \cap G \text{ the segment of }
\]
end-points $z, \xi$ meets $C$ at $f$ only. Then $v^\gamma(z)$ is bounded on $G$ provided it is bounded on $B(G)$.

Notation. If $u \neq 0 \neq v$ are complex numbers we write $\varphi u, v \equiv \arccos \left( \Re \frac{u}{v} \cdot \frac{v}{u} \right)$ for the non-oriented angle of corresponding vectors.

Let now $\psi$ be a simple path on $\langle a, \nu \rangle$. With any subdivision $D = \{a = t_0 < t_1 < \ldots < t_n = \nu\}$ $(n > 1)$ of $\langle a, \nu \rangle$ we associate the number

$$\sigma(D) = \sum_{j=1}^{n-1} \varphi \psi(t_{j+1}) - \psi(t_j), \quad \psi(t_j) - \psi(t_{j+1}) \bar{\nu}$$

and define

$$\text{bnd} \, \psi = \sup D \sigma(D).$$

Remark 5. The quantity $\text{bnd} \, \psi$, to be called the bend of $\psi$, could be defined for more general paths (cf. a number of papers by K. Iséki in Proc. Japan Academy 1961, 1962). We shall not go into details and shall only notice that simple plane paths $\psi$ with $\text{bnd} \, \psi < + \infty$ coincide with those introduced by J. Radon in connection with his investigations on the logarithmic potential (Über Randwertaufgaben beim logarithmischen Potential, Sitzber. Akad. Wiss. Wien 128 (1919), 1123-1167) and called "Kurven beschränkter Drehung" ("courbes à rotation finie" - cf. F. Riesz et B. Sz. Nagy: Léçons d'analyse fonctionnelle, chap. IV, n° 81, 91). A comparison of these paths $\psi$ with those for which $v^\psi$ is bounded is given by the following

Theorem 13. Let $\psi$ be a simple path on $\langle a, \nu \rangle, C = \psi(\langle a, \nu \rangle)$. Then

$$\sup_{x \in C} v^\gamma(x) \leq \text{bnd} \, \psi.$$
Remark 6. While
\[ \sup_{z \in \mathbb{C}} \psi(z) < +\infty \]
whenever \( \lim \psi < +\infty \), the converse is not true as simple examples show. It is also easily seen that the class of smooth paths fulfilling so-called Ljapunoff conditions (compare N.M. Gjunt'or: Teorija potënciala i jejo primëñni:ja k osnovnym zadañam matëmatišeskoj fiziki, Gostëchizdat 1953, where analogous surfaces in 3-space are considered) is included in the class of all \( \psi \) with (2). It is worth mentioning that some investigations on the logarithmic potential which are usually connected with Radon's "Kurven beschränkter Drehung" can be carried out for more general paths fulfilling (2).