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ON THE PRODUCT AND SUM OF A SYSTEM OF TRANSFORMATION SEMI-GROUPS

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1. Introduction

If X is a set, then $(F;X)$ will denote a semigroup F of transformations of X into itself. Now if a system of transformation semigroups is given, $\{(F_\alpha;X_\alpha): \alpha \in A\}$, there are several ways to construct from these a transformation semigroup F operating on the set $X = \bigcup_{\alpha \in A} X_\alpha$. We will consider two methods; as they give us essentially the direct product and the direct sum in the case that the X_α are pairwise disjunct, we call the transformation semigroups $(F;X)$, constructed from the $(F_\alpha;X_\alpha)$ by the methods considered, the product and the sum of the transformation semigroups $(F_\alpha;X_\alpha)$.

We are mainly interested in the situation when the new transformation group $(F;X)$ turns out to be commutative. In the case of the product, it is sufficient to assume that all factors $(F_\alpha;X_\alpha)$ are commutative; in the case of the sum, another condition is needed.

In the last section, the theory is applied to obtain an embedding of a given <u>commutative</u> transformation semigroup $(F;X)$ into a commutative transformation semigroup $(G;X)$ that leaves the same subsets of X invariant as F does, and that is maximal in this respect. The semigroup $(G;X)$

turns out to be uniquely determined. Then the previous results are applied to generalise a theorem on the existence of a common fixed point of a commutative system of mappings. And finally we use them to prove that every commutative semigroup is contained in a product of algebraically generated transformation semigroups.

2. Notation

If X is a non-void set, the class of all mappings $f : X \to X$ will be denoted by X^X. This is a semigroup under functional composition o :

$$(f \circ g)(x) = f(g(x))$$

for all $f, g \in X^X$ and all $x \in X$.

If F is a subsemigroup of X^X, we will often write $(F;X)$, to indicate the set transformed by the elements of F.

A system $F \subset X^X$ is called commutative if $f \circ g = g \circ f$ for all $f, g \in F$.

A subset Y of X is said to be invariant under $F \subset X^X$ if $F(Y) \subset Y$. Here $F(Y) = \{f(y) : f \in F \text{ and } y \in Y\}$. If $f \in X^X$ and $A \subset X$, then $f | A$ denotes the restriction of f to A. If $F \subset X^X$ and $A \subset X$, then $F | A = \{f | A : f \in F\}$.

If $F \subset X^X$ and $x \in X$, then $F(x)$ is called the orbit of x under F; every orbit is an invariant set.

Let $\mathcal{Y}$ be a family of subsets of a set X. A system $F \subset X^X$ is said to be $\mathcal{Y}$-invariant if every member of $\mathcal{Y}$ is an invariant set under F. The system F is called a maximal commutative $\mathcal{Y}$-invariant system if it is commutative and $\mathcal{Y}$-invariant, and if there is no commutative $\mathcal{Y}$-invariant system $G \subset X^X$ such that $F \subset G$, $F \neq G$. The system F is called a maximal commutative system if it is a maximal commutative $\{\emptyset\}$ - invariant system. Here $\emptyset$ denotes the empty set.

A maximal commutative $\mathcal{J}$-invariant system is always a commutative semigroup containing the identity mapping $i : X \to X$.

The cartesian product of sets F_α, $\alpha \in A$, is denoted by $\prod_{\alpha \in A} F_\alpha$. If $f \in \prod_{\alpha \in A} F$, then f_α denotes the component of f in F_α, and we will also write $(f_\alpha)_{\alpha \in A}$ instead of f.

3. The product of a system of transformation semigroups

In this section and in the next one we consider a family $\{(F_\alpha;X_\alpha) : \alpha \in A\}$ of transformation semigroups: A is a non-void set of indices, and $F_\alpha \subset X_\alpha^{X_\alpha}$ for each $\alpha \in A$. The identity map of X_α onto itself will be denoted by i_α; it is assumed that $i_\alpha \in F_\alpha$ for each $\alpha \in A$. The union of all sets X_α will be denoted by X:

(3.1) $$X = \bigcup_{\alpha \in A} X_\alpha ,$$

and the identity map of X onto itself will be denoted by i.

Proposition 1. Let S be the following subset of $\prod_{\alpha \in A} F_\alpha$:

(3.2) $$S = \{(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} F_\alpha : (\forall \alpha, \beta \in A)\ (f_\alpha | X_\alpha \cap X_\beta = f_\beta | X_\alpha \cap X_\beta)\}$$

Furthermore, let $F \subset X^X$ be defined in the following manner:

(3.3) $$F = \{f \in X^X : (\exists\ s \in S)\ (\forall\ \alpha \in A)\ (f | X_\alpha = s_\alpha)\} .$$

Then F is a semigroup of transformations of X into itself, containing the identity map i. If F_α is commutative for every $\alpha \in A$, then F is also commutative.

Proof.

First we show the following: if $s = (s_\alpha)_{\alpha\in A} \in S$ and $t = (t_\alpha)_{\alpha\in A} \in S$, then also $(s_\alpha \circ t_\alpha)_{\alpha\in A} \;\; S$.

As the F_α are semigroups, it is clear that $(s_\alpha \circ t_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} F_\alpha$. Now take $\alpha, \beta \in A$; we must show that

(3.4) $$s_\alpha \circ t_\alpha \mid X_\alpha \cap X_\beta = s_\beta \circ t_\beta \mid X_\alpha \cap X_\beta \, .$$

But we know that

(3.5) $$s_\alpha | X_\alpha \cap X_\beta = s_\beta \mid X_\alpha \cap X_\beta \, ,$$

(3.6) $$t_\alpha | X_\alpha \cap X_\beta = t_\beta \mid X_\alpha \cap X_\beta \, ,$$

as $s, t \in S$; this implies that $X_\alpha \cap X_\beta$ is invariant under s_α, s_β, t_α and t_β. The assertion (3.4) now follows from (3.5) and (3.6).

We now can prove that F is a semigroup. It is evident that F is non-void, as $(i_\alpha)_{\alpha\in A} \in S$, and hence $i \in F$. Take $f, g \in F$. There exist $s, t \in S$ such that for every $\alpha \in A$

(3.7) $$f | X_\alpha = s_\alpha \, , \qquad g | X_\alpha = t_\alpha \, .$$

It follows that $f(X_\alpha) \subset X_\alpha$ and $g(X_\alpha) \subset X_\alpha$; hence

(3.8) $$f \circ g | X_\alpha = s_\alpha \circ t_\alpha \, .$$

As $(s_\alpha \circ t_\alpha)_{\alpha\in A} \in S$, this shows that $f \circ g \in F$.

Finally, we assume that every F_α is commutative. Take again $f, g \in F$ and let $s, t \in S$ such that (3.7) holds. Then it follows from (3.8) that

$$f \circ g | X_\alpha = s_\alpha \circ t_\alpha = t_\alpha \circ s_\alpha = g \circ f | X_\alpha$$

for every $\alpha \in A$; hence $f \circ g = g \circ f$. Thus F is commutative.

<u>Definition 1</u>. The transformation semigroup $F \subset X^X$, defined in proposition 1 (by (3.2) and (3.3)), is called the <u>product</u>

of the transformation semigroups $(F_\alpha ; X_\alpha)$, $\alpha \in A$, and is denoted by

$$\mathop{P}_{\alpha \in A} F_\alpha \qquad \text{or} \qquad P\{F_\alpha : \alpha \in A\} .$$

It follows from the construction of $F = \mathop{P}_{\alpha \in A} F_\alpha$ that every set X_α is an invariant subset of X under F. Hence:

Proposition 2. The transformation semigroup $\mathop{P}_{\alpha \in A} F_\alpha$ is $\{X_\alpha : \alpha \in A\}$ - invariant.

Proposition 3. If the sets X_α, $\alpha \in A$, are pairwise disjoint, then the abstract semigroup $(\mathop{P}_{\alpha \in A} F, o)$ is isomorphic with the (unrestricted) direct product of the abstract semigroups (F, o).

Proof. If S and F are as in (3.2) and (3.3), then, under the assumption that the X_α are pairwise disjoint, the set S is equal to the set $\prod_{\alpha \in A} F_\alpha$. If we define a multiplication $.$ in S by

$$s \,.\, t = (s_\alpha \circ t_\alpha)_{\alpha \in A} ,$$

then $(S, .)$ is even isomorphic with the direct product of the semigroups (F_α, o). The proposition now follows from the fact that

$$(3.9) \qquad f \rightarrow (f | X_\alpha)_{\alpha \in A}$$

is an isomorphism of (F, o) onto $(S, .)$.

Proposition 4. If $X_\alpha = X$, for every $\alpha \in A$, then $\mathop{P}_{\alpha \in A} F_\alpha = \bigcap_{\alpha \in A} F_\alpha$.

Proof. If again S and F are as defined in (3.2) and (3.3), then $(f_\alpha)_{\alpha \in A} \in S$ implies

$$f_\alpha = f_\alpha | X = f_\alpha | X_\alpha \cap X_\beta = f_\beta | X_\alpha \cap X_\beta = f_\beta | X = f_\beta$$

for all $\alpha, \beta \in A$. Conversely, if $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} F_\alpha$, and $f_\alpha = f_\beta$ for all $\alpha, \beta \in A$, then $(f_\alpha)_{\alpha \in A} \in S$. This proves the assertion, as $f_\alpha = f_\beta$ for all $\alpha, \beta \in A$ implies $f_\alpha \in \bigcap_{\alpha \in A} F_\alpha$.

4. <u>The sum of a system of transformation semigroups</u>

<u>Definition 2.</u> Let $\{(F_\alpha;X_\alpha) : \alpha \in A\}$ be a system of transformation semigroups, and let $X = \bigcup_{\alpha \in A} X_\alpha$. The transformation semigroup $F \subset X^X$, generated by the set

(4.1) $T = \{f \in X^X : (\exists \alpha \in A)\ (\exists\ f_\alpha \in F_\alpha)\ (f|X_\alpha = f_\alpha$ and $f|X \setminus X_\alpha = i|X \setminus X_\alpha)\}$

is called the <u>sum</u> of the transformation semigroup $(F_\alpha;X_\alpha)$, and is denoted by

$$\mathfrak{S}_{\alpha \in A} F_\alpha \qquad \text{or } \mathfrak{S}\ \{F_\alpha : \alpha \in A\} .$$

It follows from the definition that for every $\alpha \in A$ there is an isomorphism of F_α into $\mathfrak{S}_{\beta \in A} F_\beta$.

We are mainly interested in the case that $\mathfrak{S}_{\alpha \in A} F_\alpha$ is a commutative semigroup. By the above remark, every F_α then has to be commutative. But this is not sufficient; e.g. if $X_1 = X_2 = \{0, 1\}$, and if F_1 consists only of i and the map f_1 such that $f_1(0) = f_1(1) = 0$, while F_2 consists of i and the map f_2 such that $f_2(0) = f_2(1) = 1$, then $(F_1;X_1)$ and $(F_2;X_2)$ are commutative, but $\mathfrak{S}\{F_1, F_2\}$ is not commutative.

The following condition on the family $\{(F_\alpha;X_\alpha) : \alpha \in A\}$ will turn out to be sufficient, together with the commutativity of all F_α , in order to ensure that $\mathfrak{S}_{\alpha \in A} F_\alpha$ is commutative:

(C) for all $\alpha, \beta \in A$, the sets $X_\alpha \cap X_\beta$ and $X_\alpha \setminus X_\beta$ are invariant subsets of X_α under F_α , and if $f_\alpha \in F_\alpha$ and $f_\beta \in F_\beta$, then $f_\alpha | X_\alpha \cap X_\beta$ and $f_\beta | X_\alpha \cap X_\beta$ commute.

Proposition 5. Let $\{(F_\alpha;X_\alpha) : \alpha \in A\}$ be a family of commutative transformation semigroups, each contatining the identity mapping $i_\alpha : X \to X_\alpha$, and let condition (C) be satisfied. Then $\mathop{\$}_{\alpha \in A} F_\alpha$ is a commutative transformation semigroup containing the identity map.

Proof. Let T be as in (4.1), and let F be the subsemigroup of X^X generated by T . As it is evident that $i \in F$ we have only to show that T is commutative. Let $f, g \in T$. Then there are α , $\beta \in A$ and $f_\alpha \in F_\alpha$, $f_\beta \in F_\beta$ such that

$$f|X_\alpha = f_\alpha \ ; \qquad g|X_\beta = f_\beta \ ;$$
$$f|X \setminus X_\alpha = i|X \setminus X_\alpha \ ;$$
$$g|X \setminus X_\beta = i|X \setminus X_\beta \ .$$

As condition (C) is assumed to be satisfied, $f|X_\alpha \cap X_\beta$ and $g|X_\alpha \cap X_\beta$ commute. Furthermore, $f|X \setminus (X_\alpha \cap X_\beta) =$ $= g|X \setminus (X_\alpha \cup X_\beta) = i|X \setminus (X_\alpha \cup X_\beta)$. Hence we need only check what happens with points in $X_\alpha \setminus X_\beta$ or in $X_\beta \setminus X_\alpha$. Because of the symmetry of the situation, we may restrict our attention to points in $X_\alpha \setminus X_\beta$.

Let $x \in X_\alpha \setminus X_\beta$. Then

$$(f \circ g)(x) = f(g(x)) = f(x) = f_\alpha(x) \ ;$$

as $X_\alpha \setminus X_\beta$ is supposed to be invariant under F_α , $f_\alpha(x) \in X_\alpha \setminus X_\beta$; hence

$$f_\alpha(x) = g(f_\alpha(x)) = g(f(x)) = (g \circ f)(x) \ .$$

This finishes the proof.

Proposition 6. If the sets X_α , $\alpha \in A$, are pairwise disjoint, then the abstract semigroup ($\mathop{\$}_{\alpha \in A} F_\alpha$, o) is isomorphic to the direct sum (restricted direct product) of the abstract semigroups $(F_\alpha, \circ)$, $\alpha \in A$.

Proof. Let T be defined by (4.1). Let φ be the mapping (3.9). Then maps T 1.1 onto the subset of $\prod_{\alpha \in A} F_\alpha$, consisting of all $(f_\alpha)_{\alpha \in A}$ such that $f_\alpha \neq i_\alpha$ for at most one $\alpha \in A$; and φ maps F 1.1 onto the subset of $\prod_{\alpha \in A} F_\alpha$ such that $f_\alpha \neq i_\alpha$ for only finitely many $\alpha \in A$. It is immediately seen that $\varphi | F$ is a homomorphism of (F, o) into the direct product of the (F_α, o); hence $\varphi | F$ is an isomorphism, and $\varphi(F)$ is exactly the direct sum of the (F_α, o).

Proposition 7. Assume $X_\alpha = X$, for every $\alpha \in A$. Then condition (C) is satisfied if and only if $\bigcup_{\alpha \in A} F_\alpha$ is commutative, and $\mathfrak{S}_{\alpha \in A} F_\alpha$ is the subsemigroup of X^X generated by $\bigcup_{\alpha \in A} F_\alpha$.

Proof: evident.

5. Commutative semigroups that are maximal with respect to their system of invariant sets

In this section, $(F;X)$ is a commutative transformation semigroup, containing the identity transformation, and $\mathcal{J}$ will always denote a family of subsets of X that are invariant under F.

If $\mathcal{J}$ is such a family, then $\bigcup \mathcal{J}$ will denote the set $\bigcup\{A : A \in \mathcal{J}\}$, and $\mathbb{P}(\mathcal{J})$ will denote the semigroup

$$\mathbb{P}(\mathcal{J}) = \mathbb{P}\{F|A : A \in \mathcal{J}\}.$$

The following lemma is almost obvious:

Lemma 1. $f \in \mathbb{P}(\mathcal{J}) \Longleftrightarrow f|A \in F|A$ for all $A \in \mathcal{J}$.

From this lemma, the following propositions follow without difficulty:

Proposition 8. If $\bigcup \mathcal{J} = X$, then $F \subset \mathbb{P}(\mathcal{J}) \subset X^X$.

(If $\cup \mathcal{J} \neq X$, then certainly not $F \subset \mathbb{P}(\mathcal{J})$, as $\mathbb{P}(\mathcal{J})$ consists of mappings of $\cup \mathcal{J}$ into itself.)

Proposition 9. Let both $\mathcal{J}_1$ and $\mathcal{J}_2$ consist of subsets of X that are invariant under F . If $\cup \mathcal{J}_1 = \cup \mathcal{J}_2$, then $\mathcal{J}_1 \subset \mathcal{J}_2$ implies $\mathbb{P}(\mathcal{J}_1) \supset \mathbb{P}(\mathcal{J}_2)$.

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are both families of subsets of a set X , we will say that $\mathcal{J}_1$ is a refinement of $\mathcal{J}_2$, and write

$$\mathcal{J}_1 \leqq \mathcal{J}_2 ,$$

if for every $A_1 \in \mathcal{J}_1$ there is an $A_2 \in \mathcal{J}_2$ such that $A_1 \subset A_2$.

Proposition 10. Let both $\mathcal{J}_1$ and $\mathcal{J}_2$ consist of subsets of X that are invariant under F . If $\cup \mathcal{J}_1 = \cup \mathcal{J}_2$ and $\mathcal{J}_1 \leq \mathcal{J}_2$, then $\mathbb{P}(\mathcal{J}_1 \cup \mathcal{J}_2) = \mathbb{P}(\mathcal{J}_2)$.

Proof. By proposition 9, $\mathbb{P}(\mathcal{J}_1 \cup \mathcal{J}_2) \subset \mathbb{P}(\mathcal{J}_2)$ on the other hand,

$$f \in \mathbb{P}(\mathcal{J}_2) \Longleftrightarrow (\forall A \in \mathcal{J}_2)\ (f|A \in F(A)) \Rightarrow (\forall A \in \mathcal{J}_1 \cup \mathcal{J}_2)\ (f|A \in F|A)$$
$$f \in \mathbb{P}(\mathcal{J}_1 \cup \mathcal{J}_2) .$$

Example. If $X \in \mathcal{J}$, then $\mathbb{P}(\mathcal{J}) = F$.

Remark. If A is not an invariant subset of X , then $F|A$ is not a semigroup. However, if we define $F \| A = \{f|A : f \in F$ and $f(A) \subset A\}$ then $F \| A$ is a semigroup under composition. It is seen at once that

$$\mathbb{P}\{(F;X), (F \| A;A)\} = \{f \in F : fA \subset A\} ;$$

hence if A is not invariant, $F \not\subset \mathbb{P}(F, F \| A)$, although of course $X \cup A = X$.

Lemma 2. Let $\mathcal{J}_1$ be the class of all subsets of X that are invariant under F , and let $\mathcal{J}_2$ be the class of all orbits under F , and let $F \subset G \subset X^X$.

Then G is a commutative $\mathcal{J}_1$-invariant system if and only if G is a commutative $\mathcal{J}_2$-invariant system.

Proof. As $\mathcal{J}_2 \subset \mathcal{J}_1$, every $\mathcal{J}_1$-invariant system is $\mathcal{J}_2$ invariant. On the other hand, if $A \in \mathcal{J}_1$, then

$$A = F(A) = \bigcup \{ F(x) : x \in A\} = \bigcup \{B \in \mathcal{J}_2 : B \subset A\} .$$

Hence every $\mathcal{J}_2$-invariant system is $\mathcal{J}_1$-invariant.

Lemma 3. Let $G \subset X^X$ be commutative. If there exists an $e \in X$ such that $G(e) = X$, then G is a maximal commutative semigroup.

Proof. Let $f \in X^X$ such that f commutes with every $g \in G$. We will show that $f \in G$. As $G(e) = X$, there exists a $g_0 \in G$ such that $f(e) = g_0(e)$. Let x be an arbitrary element of X; then there is a $g \in G$ such that $g(e) = x$, and it follows that

$$f(x) = f \circ g(e) = g \circ f(e) = g \circ g_0(e) = g_0 \circ g(e) = g_0(x) .$$

Hence $f = g_0 \in G$.

In particular, we have the following:

Lemma 4. If $F \subset X^X$ is a commutative semigroup, containing the identity map, then for every orbit $F(x)$ under F, $F|F(x)$ is a maximal commutative semigroup of mappings $F(x) \to F(x)$.

Theorem 1. Let $F \subset X^X$ be a commutative semigroup, containing the identity map. Let $\mathcal{J}$ be the class of all subsets of X that are invariant under F. Then there exists one and only one maximal commutative $\mathcal{J}$-invariant semigroup $G \subset X^X$ containing F; and

$$G = \mathbb{P} \{F|F(x) : x \in X\} .$$

Proof. Let g be any mapping $X \to X$ that commutes with every $f \in F$ and that maps every $A \in \mathcal{J}$ into itself. We will show that $g \in G$.

Take any $x \in X$. Then $g|F(x)$ maps $F(x)$ into itself, as $F(x) \in \mathcal{J}$, and $g|F(x)$ commutes with every mapping in $F|F(x)$. But by lemma 4, $F|F(x)$ is a maximal commutative semigroup; hence $g|F(x) \in F|F(x)$. It now follows from lemma 1 that $g \in G$.

An immediate consequence is that $F \subset G$ (this also follows from proposition 8). So it remains only to be proved that G is $\mathcal{J}$-invariant. But by proposition 2, G is $\mathcal{J}_2$-invariant, where $\mathcal{J}_2 = \{F(x) : x \in X\}$; now apply lemma 2.

<u>Corollary</u>: If $F \subset X^X$ is a maximal commutative transformation semigroup, then

$$F = \mathbf{P}\{F|F(x) : x \in X^X\} .$$

A family of orbits $\{F(x) : x \in Y\}$, where Y is a subset of X, is called an <u>F-orbit cover</u>, or shortly an <u>F-cover</u> of X, if $F(Y) = X$.

From proposition 10 and theorem 1 we deduce at once:

<u>Theorem 2</u>. If $\{F(x) : x \in Y\}$ is an F-cover of X, then $\mathbf{P}\{F|F(x) : x \in Y\}$ is the maximal commutative $\mathcal{J}$-invariant semigroup containing F (where $\mathcal{J}$ is the family of all subsets of X that are invariant under F).

In [1] the following theorem was proved ([1], Theorem 1):

"Let F be a maximal commutative semigroup of mappings of a set X into itself, and let $r(F) \neq \emptyset$. If each $f \in F$ has a fixed point, then all mappings in F have precisely one common fixed point."

Here $r(F) = \{f \in F : (\forall f_1 \in F)\ (\exists f_2 \in F)\ (f = f_1 \circ f_2)\}$ is the set of all mappings $f \in F$ that are common multiples of all mappings in F .

Using the concepts developed in this paper, we may generalise this theorem as follows:

Theorem 3. Let $F \subset X^X$ be a maximal commutative $\mathcal{J}$-invariant transformation semigroup (where $\mathcal{J}$ again is the family of all subsets of X that are invariant under F). If $r(F) \neq \emptyset$ and if each $f \in F$ has a fixed point, then all mappings in F have a common fixed point.

The proof is exactly the same as the first part of the proof of [1], Theorem 1 . It is easily seen that the mapping g , constructed in [1], leaves all sets of $\mathcal{J}$ invariant; hence the weaker assumption that F is maximally $\mathcal{J}$-invariant suffices in order to conclude that $g \in F$.

Finally we will give one more application of the above product construction. In order to do so, however, we need the concept of an algebraically generated transformation semigroup.

Take an abstract semigroup $(X; .)$ and consider all left multiplications in X , i.e. all mappings f_a, $a \in X$, defined by

(5.1) $$f_a(x) = a.x \ .$$

These mappings constitute a semigroup $F \subset X^X$. If X has an identity element, it is even true that the abstract semigroup $(F; o)$ is isomorphic with $(X; .)$. (In fact, in that case the correspondence $a \to f_a$ is an isomorphism of $(X; .)$ onto $(F; o)$.) Now transformation semigroups of this kind will be called algebraically generated. More explicitly:

Definition 3. A transformation semigroup $F \subset X^X$ is called algebraically generated if there exists a binary operation . om X such that

(i) $(X; .)$ is a semigroup with unit;

(ii) $F = \{f_a : a \in X\}$, where f_a is as defined in (5.1).

Using lemma 3, it is easy to give a complete characterisation of all commutative transformation semigroups that are algebraically generated.

<u>Lemma 5</u>. A commutative transformation semigroup $F \subset X^X$ is algebraically generated if and only if there exists an $e \in X$ such that $F(e) = X$.

<u>Proof</u>.

First assume F to be algebraically generated, say by the semigroup structure $(X; .)$. Then if e is the unit element of $(X; .)$, it is immediate that $F(e) = X$.

Conversely, assume $F(e) = X$, for some $e \in X$. From the proof of lemma 3 it follows at once that the mapping $\varphi : F \to X$, defined by

$$\varphi(f) = f(e)$$

is a 1-1-mapping of F onto X. Define a binary operation $.$ in X by

$$x . y = \varphi(\varphi^{-1}(x) \circ \varphi^{-1}(y)) .$$

Then $(X; .)$ is a commutative semigroup, with e as unit element and $F = \{f_a : a \in X\}$, as

$f_a(x) = a.x = \varphi(\varphi^{-1}(a) \circ \varphi^{-1}(x)) = (\varphi^{-1}(a) \circ \varphi^{-1}(x))(e) =$
$= \varphi^{-1}(a)(x)$.

We now prove the following theorem, which states in effect that every commutative transformation semigroup can be built up, using the product construction of section 3, from algebraically generated semigroups:

<u>Theorem 4</u>. Let $F \subset X^X$ be a commutative transformation

semigroup, and let m be the cardinal number of an F-cover of X . Then F is a subsemigroup of a product of m algebraically generated commutative semigroups.

Proof.

Theorem 2 assers that F is a subsemigroup of a product of m semigroups $F|F(x)$, and lemma 5 shows that all these semigroups are algebraically generated (as $(F|F(x))(x) = F(x)$) .

References:

[1]. Z. HEDRLÍN, Two theorems concerning common fixed points of commutative mappings, CMUC, 3,2 (1962).