Cor P. Baayen; Zdeněk Hedrlín
Commutative polynomial semigroups on a segment

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1. Introduction

A commutative semigroup of mappings of a set $X$ is a family of mappings $X \to X$ which is a commutative semigroup under composition of functions. A commutative polynomial semigroup of mappings of a subset $X$ of the real line $R$ (shortly: an $X$-cps) is a commutative semigroup of mappings $X \to X$, all elements of which are restrictions to $X$ of (real) polynomials on $R$. Such a semigroup $S$ is called maximal if every continuous map $g : X \to X$ which commutes with all $f \in S$ itself belongs to $S$, and entire if it contains (restrictions to $X$ of) polynomials of every non-negative degree.

If $S_1$ is a semigroup of continuous maps $X_1 \to X_1$ ($i = 1, 2$), and if $\tau$ is a homeomorphism of $X_1$ onto $X_2$ such that $S_2 = \{ \tau \circ f \circ \tau^{-1} \mid f \in S_1 \}$, then $S_1$ and $S_2$ are called equivalent (by means of $\tau$). In that case the transformation $f \mapsto \tau \circ f \circ \tau^{-1}$ is an isomorphism of the abstract semigroup $S_1$ onto the abstract semigroup $S_2$.

In this note we determine, up to equivalence, all entire $I$-cps, where $I$ is the closed unit segment $[0, 1]$. Moreover, we establish which of these $I$-cps are maximal and which not. We denote by $J$ the segment $[-1, 1]$.

2. Commutative polynomial semigroups of mappings $R \to R$

and $J \to J$.

It follows from results of J.F. Ritt [7, 8] and of H.D. - 173 -
Block and H.P. Thielman [5] that every entire R-cps is equivalent by means of a linear transformation to one of the following three semigroups of polynomials:

(i) the semigroup $P$, consisting of the maps

$$P_n(x) = x^n;$$

(ii) the semigroup $P^*$, consisting of all $P_n$, $n > 1$ and the map $P_0$ such that

$$P_0(x) = 0$$

for all $x$;

(iii) the semigroup $T$ of all Chebyshev polynomials

$$T_0, T_1, T_2, \ldots,$$

where

$$T_n(x) = \cos(n \cdot \arccos x).$$

The first two semigroups are not maximal; e.g. consider $x^3$.

Lemma 1. There exists a unique maximal commutative semigroup $\overline{P}(\overline{P}^*)$ of continuous maps $R \to R$ containing $P$ ($P^*$, respectively). The semigroup $\overline{P}(\overline{P}^*)$ consists of the following maps: all maps $x \to |x|^\varepsilon$, $\varepsilon > 0$ a real number; all maps $x \to |x|^\varepsilon$ sign $x$, $\varepsilon > 0$ a real number; and all maps in $P$ ($P^*$, respectively).

Proof. It is immediately verified that $\overline{P}$ and $\overline{P}^*$ are commutative semigroups. In order to show their maximality, and the fact that they are the only maximal semigroups containing $\overline{P}$ or $\overline{P}^*$, we proceed as follows.

Let $f$ be any continuous map $R \to R$ commuting with all maps in $P$ or in $P^*$. Take any $a$ with $0 < a < 1$ and let $f(a) = \alpha$. As $\alpha = P_2 f(\sqrt{a})$, $\alpha > 0$ if $\alpha = 0$, it follows that $f(x^2) = \alpha^2 = 0$ for all rational $r$, because $f \circ P_n = P_n \circ f$ for all natural $n$. Hence $f(x) = 0$ for $x > 0$; if $x \leq 0$, $P_2 f(x) = f(x^2) = 0$ implies again $f(x) = 0$. Thus $f$ is identically zero.
Assume $\alpha > 0$ and let $\xi \in \mathbb{R}$ with $\alpha^\xi = \alpha$. Then as $f$ and $P_n$ commute, $f(a^r) = a^{r\xi}$ for all rational $r$; hence $f(x) = x^\xi$ for $x > 0$. If $x < 0$, then $P_2f(x) = fP_2(x) = (x^2)^\xi$, hence $f(x) = \pm |x|^\xi$. As $f$ is continuous, the lemma follows.

The situation is different for the semigroup $T$: this semigroup is maximal. In order to show this, we consider the following mappings of the unit interval $I$ into itself, first introduced in [2]:

$$t_0(x) = 0 \text{ for all } x;$$

and, if $n > 1$:

$$\begin{cases}
t_n\left(\frac{2k}{n}\right) = 0, & t_n\left(\frac{2k+1}{n}\right) = 1 \quad (k = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor); \\
t_n \left[ \frac{k}{n}, \frac{k+1}{n} \right] \text{ is linear } (k = 0, 1, 2, \ldots, n-1).
\end{cases}$$

These so-called multihits are easily seen to constitute a commutative semigroup $M$; in fact, $t_n \circ t_m = t_{n+m}$. In [2] P.C. Baayen, W. Kuyk and M.A. Maurice proved much more: the semigroup of all $t_n$, $n = 0, 1, 2, \ldots$, is a maximal commutative semigroup of continuous maps $I \to I$.

**Lemma 2.** The semigroup $M$ is equivalent to the semigroup $T'$ of all Chebyshev polynomials $T_n$, restricted to the segment $J$, by means of the homeomorphism $\tau: [0,1] \to [-1, 1]$ such that

$$\tau x = \cos \pi x.$$

**Proof:** immediate.

Hence we have shown:

**Lemma 3.** The $J$-cops $T$ is maximal.

This strengthens considerably a result of G. Baxter and J.T. Joichi [3], who showed that $T$ cannot be embedded in a 1-parameter semigroup of commuting functions.

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We conclude this section with a triviality.

**Lemma 4.** Let $Q_1, Q_2$ be polynomials commuting on some non-degenerate segment. Then $Q_1$ and $Q_2$ commute everywhere on $\mathbb{R}$.

3. **Commutative polynomial semigroups of mappings $I \to I$**

It follows from the results of section 2 that every entire $I$-cp is equivalent by means of a linear transformation to a semigroup $S|A$, where $S$ is one of the $R$-cps $T, P, P^*$ and $A$ is a closed segment $[a, b], a < b$, that is invariant under $S$.

The only non-degenerate segment mapped into itself by $T$ is $[-1, +1]$. The only non-trivial segments mapped into themselves by $P$ are the segments $[-a, 1]$, with $0 < a < 1$; we write $P(a)$ for the $[-a, 1]$-cp of all $P_n|[-a, 1]$, $n = 0, 1, 2, \ldots$. The only non-trivial segments invariant under $P^*$ are the segments $[-a, b]$, with $0 < a < 1$, $a^2 < b < 1$, $b \neq 0$; we write $P^*(a, b)$ for the $[-a, b]$-cp of all $P_n|[-a, b]$, $n \geq 1$ together with $P_0^*|[-a, b]$.

**Lemma 5.** Each of the semigroups $P(a)$, $0 < a < 1$, is not maximal, and is contained in a unique maximal $[-a, 1]$-semigroup $\overline{P(a)}$. Similarly each $P^*(a, b)$ is contained in a unique maximal $[-a, b]$-semigroup $\overline{P^*(a, b)}$.

**Proof.** In the same way as in the proof of Lemma 1 one shows that $\overline{P(a)} = P|[-a, 1]$ is the unique maximal commutative semigroup of continuous maps $[-a, 1] \to [-a, 1]$ containing $P(a)$. Similarly $\overline{P^*(a, b)} = P^*|[-a, b]$.

**Remark:** If $S$ is a semigroup of mappings of a set $X$ into itself, and if $A \subset X$, then $S|A$ denotes the semigroups of mappings of $A$ into itself, consisting of all mappings $f|A$ such that $f \in S$ and $f(A) \subset A$ (cf. [6]).
Theorem 1. There are two maximal entire I-cps; they are both equivalent to $T^*$ (or to $M$).

Proof. Every maximal entire I-cps must be equivalent by means of a linear map to $T' = T|[0,1]$. There exist two linear maps of $[-1, +1]$ onto $I = [0,1]$.

Lemma 6. If $0 < a, b < 1$, then $P(a)$ and $P(b)$ are equivalent by means of the homeomorphism $\tau$,

$$\tau(x) = \frac{\log x}{\log a} \cdot |x|^\varepsilon,$$

where $\varepsilon = \frac{\log b}{\log a}$.

Lemma 7. Let $0 \leq a_1 \leq 1, a_2^2 \leq b_1 < 1, b_1 \neq 0 (i = 1, 2)$. The semigroups $P^*(a_1, b_1)$ and $P^*(a_2, b_2)$ are equivalent if and only if there exists a real number $\varepsilon > 0$ such that $a_2 = a_1^\varepsilon, b_2 = b_1^\varepsilon$.

Proof. Suppose $P^*(a_1, b_1)$ and $P^*(a_2, b_2)$ are equivalent by means of $\tau$. Then we have, for arbitrary $x \in [-a_1, b_1]$ and for arbitrary integers $n \geq 1$, that $P_n(x) = (\tau^{-1} \circ P_n \circ \tau)(x)$; i.e. $(\tau \circ P_n)(x) = (P_n \circ \tau)(x)$. It follows (cf. lemma 1) that either $\tau$ is of the form: $\tau(x) = |x|^\varepsilon$, for all $x \in [-a_1, b_1]$, where $\varepsilon$ is some real number -- as $\tau$ is a homeomorphism this is only possible if $a_1 = 0$ -- or $\tau$ is of the form: $\tau(x) = |x|^\varepsilon \cdot \text{sign} x$. As clearly we must have: $\tau(a_1) = a_2, \tau(b_1) = b_2$, the assertion follows.

The next lemma is easily proved:

Lemma 8. No semigroup $P(a)$ is equivalent to a semigroup $P^*(b, c)$.

Consequently we have:

Theorem 2. There are infinitely many non-equivalent non-maximal entire I-cps. Each of them is equivalent to one of the following semigroups, which are all mutually inequivalent: $P(0), \ldots$
Theorem 3. Every entire I-cps is contained in a unique maximal commutative semigroup of continuous maps $I \rightarrow I$. Two entire I-cps are equivalent if and only if the maximal commutative semigroups in which they are contained are equivalent.

4. Remark on mappings commuting with $T_n$ or $P_n$, $n \geq 2$.

It was shown by P.C. Baayen and W. Kuyk in [1] that every open map of $I$ into itself that commutes with $t_2$ is itself a multihat $t_n$. From this it follows almost at once that every continuous map commuting with $t_2$ is either a $t_n$ or is everywhere oscillating (nowhere monotone).

This result has been improved very much by G. Baxter and J.T. Joichi [4], who showed the following theorem:

If a continuous map $f : I \rightarrow I$ commutes with some multihat $t_n$, $n \geq 2$, it is itself either a hat-function or a constant map.

Now we saw in section 2 that the semigroup $M$ of all hats $t_n$ is equivalent to the semigroup $T'$ of all Chebyshev polynomials on $[-1, +1]$.

Hence we conclude:

Theorem 4. Every non-constant continuous map of $[-1, +1]$ into itself that commutes with a Chebyshev polynomial $T_n$ with $n \geq 2$, is itself a Chebyshev polynomial.

For the maps $P_n$, $n \geq 2$, the situation is completely different. Consider e.g. continuous maps of $[0, 1]$ into itself which commute with $P_2$ on that interval.

There exist multitudes of such functions. For let $0 < a < 1$, and let $f_0$ be any continuous function of $[a^2, a]$ into $(0, 1)$ such that $[f_0(a)]^2 = f_0(a^2)$. If we define: $f(0) = 0$, $f(1) = 1$, $f(x) = [f_0(x^{2^n})]^{2^n}$ if $x \in [a^{2n+1}, a^{2n}]$.
(n integer), \( f \) will be a continuous map \( I \rightarrow I \) commuting with \( P^2 \).

References


