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# HOMOLOGICAL FIXED POINT THEOREMS, II.

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This paper consists of some notes and generalisations of results of the preceding paper [4].

The first of these concerns lemma 2 of [4], stating that the invariant $j$ of endomorphisms $f$ of a group $G$ is independent of the behaviour of $f$ on the periodic part of $G$. Here we present a considerably stronger result in theorem 1.

The second extends a result of [4] (for a continuous $f : S^{2n} \to S^{2n}$, $f^2$ has a fixed point) to a more general class of spaces, admitting formation of cartesian products; lemma 1 and theorem 2.

The rôle which even-dimensionality plays in this result suggests the possibility of a connection with other familiar theorems having similar restrictions: Brouwer's theorem on antipodals [1, ch. XVI, § 5], or the "hedgehog theorem" of Poincaré (loc.cit., there is no nonzero tangent vector field on $S^{2n}$). A closer examination reveals that the resemblance is only superficial: the latter theorems admit a natural generalisation to e.g. odd-dimensional spheres, as will be shown in theorem 3; our result does not.

As in [4], we consider the category $\mathcal{G}_J$ consisting of abelian groups with an integrity domain $J$ as left operators, and of their operator homomorphisms. The reader is first referred to [2], exercises D in chap.IV. There it is shown how one may assign to each group G in $\mathcal{G}_J$ a vector space $G^\wedge$ over $\hat{J}$, the quotient field of J ; and to each $f : G \to G'$ in $\mathcal{G}_J$ a $\hat{J}$-homomorphism $f^\wedge : G^\wedge \to G'^\wedge$ . The resulting object turns out to be an additive exact covariant functor $\wedge$ from $\mathcal{G}_J$ to $\mathcal{G}_{\hat{J}}$ . (The definition loc.cit. of the transitive relation $\sim$ should, however, be corrected to: $[\theta_1, x_1] \sim [\theta_2, x_2]$ iff $\theta\theta_2 x_1 = \theta\theta_1 x_2$ for some $\theta \neq 0$ in J .) The circumflex $\wedge$ will henceforth be used in this sense, and not in that of [4].

Exactness of $\wedge$ then implies that, on the category $\partial\mathcal{G}_J$ of differential groups over J , the homology functor and $\wedge$ commute:

$$H(G^\wedge) = (H(G))^\wedge \quad , \quad (f^\wedge)_* = (f_*)^\wedge \quad .$$

It is noted (loc.cit.) that $\wedge$ preserves ranks. Since $j(id_G) = (\text{rank } G)/(1 - \lambda)$ [4, section 1] , this is the f = = identity special case of the following

<u>Theorem 1</u>. If $f : G \to G$ in $\mathcal{G}_J$ , then $j(f) = j(f^\wedge)$ .

By [4, definition 3], gli depends on j ; thus theorem 1 implies $gli(f) = gli(f^\wedge)$ for $f : G \to G$ in the category of group sequences. In [4, theorem 3] it was shown that gli(f) = = $gli(f_*)$ for $f : G \to G$ in the category of differential group sequences (i.e., complexes); our present result yields, then,

$$gli(f) = gli(f_*^\wedge)$$

Proof of theorem 1. There is a canonic mapping $c : G \to \hat{G}$ defined by $c\,x = (1, x)$ ; we have $c \in Hom_J(G, G^\wedge)$ and $c\,f = f^\wedge c$ for $f \in Hom_J(G, G)$ . It is easily shown that, if B is a w-base in G [4, section 1], then c(B) is linearly independent and generates $\hat{G}$ ; thus $c(B)$ is a base in $\hat{G}$ . The relations

$$\theta_i \, f \, x_i = \Sigma_j \, \alpha_{ij} \, x_j$$

used to define matrices D, A and then p, j [4, def.1 and 2] carry over to

$$\theta_i \, f^{\wedge} \, c \, x_i = \Sigma_j \, \alpha_{ij} \, c \, x_j \, ;$$

thus they define the same matrices D, A and hence also p, j . This completes the proof.

Definition. *A triangulable space will be called non-odd if all its odd-dimensional homology groups* (over integer coefficients) *are periodic.*

This definition is a modification of an earlier inadequate version; the present formulation and also the proof of the lemma to follow were suggested to the author by Mr. A. Pultr, the referee.

Cells and even-dimensional spheres are non-odd, since their odd-dimensional homology groups are trivial. Even-dimensional projective spaces are non-odd, as may be shown directly. We note that the Euler characteristic of a non-odd space reduces to the sum of ranks of the even-dimensional homology groups; hence it is positive unless the spaces is empty.

Lemma 1. *The cartesian product of two non-odd spaces is non-odd.*

Proof. Let X, Y be non-odd; the Künneth formula (e.g. [3 , chap.I, th. 5.5.2]) is

$$H_n(X \times Y) = \sum_{p+q=n} H_p(X) \otimes H_q(Y) + \sum_{p+q=n-1} Tor(H_p(X), H_q(Y)) \, .$$

The second sum is always a periodic group; consider any summand in the first sum. For odd $n = p + q$ , one of p, q is also odd, so that by assumption one factor is periodic; hence $H_p(X) \otimes H_q(Y)$ is periodic. This completes the proof.

It may be remarked that if the condition in the definition is strengthened to "all odd-dimensional homology groups are trivial", then the corresponding lemma no longer holds.

Theorem 2. Let $f : X \longrightarrow X$ be a continuous mapping of a non-odd space $X \neq \emptyset$ . Then one of

$$f, f^2, f^3, \ldots, f^{\chi(X)}$$

has a fixed point.

As a trivial but weird example, for every map $f$ of a finite set of $n$ points into itself, at least one of $f, \ldots, f^n$ has a fixed point; this is easily checked directly, and in general, $f^n$ cannot be replaced by a preceding iterate.

Corollary. $f^{\chi(X)!}$ has a fixed point.

This includes our corollary to theorem 5 in [4], and also the Brouwer fixed - point theorem; $\chi(S^{2n}) = 2$ , $\chi(E^n) = 1$ respectively. As concerns the conjecture in [4, section 3], we now have the following result. A space X consisting of the product of n cells and m even-dimensional spheres has as Euler characteristic $\chi(X)$ the product of characteristics of its factors, namely $2^m$ . Thus one of

$$f, f^2, f^3, \ldots, f^{2^m}$$

does have a fixed point, but here one may not omit the $f^i$ with $i \neq 2^j$ (e.g. for $X = S^0 \times S^0$ ).

Proof of the theorem 2. Let $f_*$ be the homomorphism of the homology sequence of X , induced by f . From theorem 1 ,

$$\text{gli}(f) = \text{gli}(f_*^{\wedge}) .$$

From [4], section 3, lemma 4 and definition 3, we then have for the Lefschetz number J

(2) $$J(f^r) = J(f_*^{\wedge r}) = \sum_q \text{tr}\,(f_{*2q}^{\wedge r})$$

since by our assumption on X, $H_q(X)^\wedge = 0$ for odd dimensions q . Finally, from the proof of theorem 2 in [4],

$$\text{(3)} \qquad tr(f_{*2q}^{\wedge r}) = \sum_{j=1}^{r_q} \lambda_{j,q}^{r}$$

where $r_q$ = rank $H_q(X)^{\wedge}$ = rank $H_q(X)$ , and $\lambda_{j,q}$ are certain complex numbers (characteristic roots of certain matrices $D_{2q}^{-1} A_{2q}$). It is known that $tr\,(f_{*o}^{\wedge r}) = 1$ if X is connected (e.g.[1], chap. XVII, § 1); in our case we have at least that $tr(f_{*o}^{\wedge r}) = m$ , a <u>positive</u> integer since X is nonempty.

Substitute (3) into (2), omit all $\lambda = 0$ , assemble all $\lambda = 1$ , and finally all equal $\lambda$'s . Thus we may write

$$J(f^r) = m_o + \sum_{j=1}^{\chi(X)-1} m_j \lambda_j^r$$

with $m_j \geq 0$ integers, $m_o > 0$ , $\lambda_j$'s distinct with $0 \neq \lambda_j \neq 1$ . (By non-oddness, $\chi(X) = \sum$ rank $H_{2q} = \sum r_{2q}$ ; thus there are at most $\chi(X)$ distinct $\lambda_j$ , of which at least one is included in the $m_o$ term.

With notation thus established, assume that the assertion of the theorem does not hold. Thus the iterate $f^r$ with $1 \leq r \leq \chi(X)$ has no fixed points, and from the Hopf-Lefschetz theorem we obtain $\chi(T)$ equations $J(f^r) = 0$ . Substracting the r-th from the following there results

$$\sum_{j=1}^{(X)-1} m_j \lambda_j^{r}(\lambda_j - 1) = 0 \qquad (1 \leq r \leq \chi(X) - 1) .$$

Consider these as a system of equations in unknowns $m_j$ . Obviously the determinant of the system is

$$\Delta = \prod_j \lambda_j \times \prod_j (\lambda_j - 1) \times V(\dots \lambda_j \dots)$$

with V the Vandermonde determinant. Since by construction the $\lambda_j$ are all distinct and $0 \neq \lambda_j \neq 1$ , we conclude $m_j = 0$ for all j . Thus our relations $J(f^r) = 0$ reduce to $m_o = 0$ ; this contradiction with $m_o > 0$ proves our theorem.

To unburden the formulation of the theorem to follow, we first introduce, provisionally, two new terms.

A topological space $T$ may be called *sphere-like* if it is triangulable connected, with positive dimension $n$, and

$$H_q(T) = 0 \quad \text{for} \quad 0 < q < n , \quad \text{rank}\, H_n(T) = 1 .$$

Obviously, spheres are sphere-like; however $S^0$ and e.g. $S^n \times S^m$ or $E^n$ are not $(n > 0)$.

A homeomorphism $h : T \to T$ of a sphere-like space will be called <u>positive</u> or <u>negative</u> in accordance with the sign of its degree. This latter term may be introduced for continuous maps $f : T \to T$ (sphere-like) as follows. Take any element $x \in H_n(T)$ of infinite order; since $H_n(T)$ has rank 1, there exists integers $\theta \neq 0$ and  such that

$$\theta f_{*n} x = \alpha x ;$$

then set

$$\text{degree}\ (f) = \frac{\alpha}{\theta} .$$

This is easily shown to be independent of the choice of $x$, $\theta$, $\alpha$. (In the notation of [4], degree $(f) = tr(f_{*n}) = \frac{d}{d\lambda} j_n(f;\lambda)|_{\lambda=0}$.) If $T = S^n$, $H_n(S^n)$ is infinite cyclic and degree $(f)$ is an integer, and coincides with the customary concept. If $f$ is a homeomorphism, degree $(f) = \pm 1$ ; for $T$ a simply connected region in $S^2$ this coincides with the sign of $f$ as defined in [4, p. 433]. The identity map is a positive homeomorphism; if $T = S^n$, then change of sign of $k$ of the $n + 1$ coordinates is positive or negative according as $k$ is even or odd.

<u>Theorem 3</u>. Let $f : T \to T$ be a continuous map of a sphere-like space $T$. Then

$$f x = h x$$

is solvable in $T$, either for all positive or for all negative homeomorphisms $h : T \to T$. If $f$ itself is a homeomorphism, then precisely one of these alternatives holds.

Proof. With the Hopf-Lefschetz theorem, the proof is almost trivial: it sufficed to consider existence of fixed points of $h^{-1} f$, and

$$J(h^{-1}f) = 1 + (-1)^n \text{ degree } (h^{-1}f) =$$
$$= 1 + (-1)^n \text{ degree } (h) \text{ degree } (f) \neq 0$$

for at least one of degree $(h) = \pm 1$. If also degree $(f) = \pm 1$, then there is precisely one possibility.

As an example, take $T = S^{2n}$. Then either $fx = x$ is solvable ( $h$ = identity, degree $(h) = 1$ ) or $fx = -x$ is solvable ($h x = -x$, degree $(h) = (-1)^{2n+1} = -1$). This is Brouwer's theorem on antipodals.

Theorem 3 suggests that it may be interesting to obtain further results on solvability of

$$f x = g x$$

for given continuous $f, g : X \to X'$.

A problem was formulated in [4], to prove

(4) $$J(f) = \chi(A)$$

for all maps $f : X \to X$ of a triangulable space $X$ and with $A$ the set of fixed points of $f$. A class of maps was exhibited for which the stronger relation

$$\text{gli}(f) = \frac{\chi(A)}{1 - \lambda}$$

holds [3, theorem 6]. The desirability of formula (4) follows from the information concerning $A$ which could be obtained from rather superficial information about $f$; e.g., the Hopf-Lefschetz fixed point theorem would follow.

However, the conjecture is not valid, and the heuristics which led to it were not sufficiently careful: there is a simple counter-example. Take $X = S^1$, treated as the unit circle in the complex plane. Let $f$ be defined by $f x = x^d$, $d$ integer. Then $f$ has degree $d$ [ 2, ch.XI, theorem 4.5], and thus

$$J(f) = 1 - d .$$

For $d \neq 1$, $f$ has precisely $|d - 1|$ fixed points, and in any case

$$\chi(A) = |d - 1|$$

for the set $A$ of fixed points of $f$. Thus $J(f) \neq \chi(A)$ for $d > 1$.

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