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BETTI NUMBERS OF REGIONS OF ATTRACTION

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Summary: it is shown that in a dynamical system, under certain conditions, the Betti numbers of a limit set coincide with those of regions of uniform attraction.

Let  $P$  be a topological space; a (global) dynamical system on  $P$  (cf. [3, chap. V], this journal, p. 121) is a map  $\tau$  with the following properties:

1°  $\tau : P \times E^1 \rightarrow P$  is continuous (the value of  $\tau$  at  $(x, \theta)$  will be denoted by  $x \tau \theta$ ),

2°  $x \tau 0 = x$ ,

3°  $(x \tau \theta_1) \tau \theta_2 = x \tau (\theta_1 + \theta_2)$ .

For fixed  $\theta \in E^1$ , define continuous maps  $t_\theta : P \rightarrow P$  by

$$t_\theta x = x \tau \theta .$$

From 2°,  $t_0$  is the identity map; from 3°,

$$t_\theta t_\sigma = t_{\theta+\sigma} ;$$

in fact, it is obvious that  $\{t_\theta\}_{\theta \in E^1}$  form a continuous abelian group of homeomorphisms  $P \approx P$ .

Further terminology: A trajectory (through  $x \in P$ ) is a subset of  $P$  of the form

$$\{x \tau \theta : \theta \in E^1\} ;$$

a critical point is a common fixed point of all  $t_\theta$ , i.e. a singleton trajectory; a cycle is a non-singleton trajectory through a fixed point of some  $t_\theta$  with  $\theta \neq 0$  /  $\theta$  is then termed

a period of the cycle); a subset  $X \subset P$  is (+)-invariant if  $t_\theta X \subset X$  for all  $\theta \geq 0$ .

Proposition 1. Each  $t_\theta$  is homotopic to the identity map, via the homotopy  $h : P \times \langle 0, 1 \rangle \rightarrow P$ ,  

$$h(x, \lambda) = x \tau \lambda \theta .$$

This result was exploited in [this journal, pp. 123-4] to obtain conditions for existence of critical points. It may be noticed that  $3^0$  is not used at all, so that stronger results may be expected.

A second proposition we shall reproduce here was obtained in [2, theorem 6]; the  $j_q$ -characteristic is defined there;  $\pi_q$  will denote the  $q$ -th Betti number,  $\chi$  the Euler characteristic.

Proposition 2. Let  $f : X \rightarrow X$  be a continuous map of a triangulable space  $X$ , let the iterates of  $f$  converge,

$$f^n \rightarrow f^\infty \text{ uniformly as } n \rightarrow \infty .$$

Then, if  $Y = f^\infty(X)$  is triangulable,

$$j_q(f^m) = \frac{\pi_q(Y)}{1 - \lambda}$$

for all  $q$  and  $1 \leq m \leq \infty$ .

Theorem. Given a dynamical system on a topological space  $P$ . Let  $\theta > 0$ , let  $X \subset P$  be (+)-invariant and such  $t_\theta, t_{2\theta}, t_{3\theta}, \dots$  converge uniformly on  $X$ . Then, setting  $X_\infty = \bigcap_{n \geq 1} t_{n\theta}(X)$ ,

$$\pi_q(X) = \pi_q(X_\infty) \text{ for all } q$$

if both  $X, X_\infty$  are triangulable.

Proof. Note first that  $t_{n\theta} = t_\theta^n$ , the  $n$ -th iterate of  $t_\theta$ ; and that  $X_\infty$  is the image of  $X$  under  $\lim t_{n\theta}$ . From proposition 2,

$$\frac{\pi_q(Y)}{1 - \lambda} = j_q(t_\theta)$$

for all  $q$ .

Homotopic maps have coinciding homologues, and hence coinciding  $j_q$ -characteristics; thus from proposition 1,

$$j_q(t_\Theta) = j_q(\text{id}_X) = \frac{\pi_q(X)}{1-\lambda}$$

for all  $q$ . The asserted formula now follows from the two exhibited relations.

Corollary 1. With the assumptions of the theorem,  $\chi(X) = \chi(X_\infty)$ . (This follows from  $\chi = \sum_q (-1)^q \pi_q$ .)

Corollary 2. Let  $x_0$  be a critical point of a dynamical system on  $P$ . Then, for any triangulable (+)-invariant set  $X \subset P$  such that

$x \tau \Theta \rightarrow x_0$  for  $\Theta \rightarrow +\infty$ , uniformly for  $x \in X$ , there hold

$$\pi_0(X) = 1, \quad \pi_q(X) = 0 \quad \text{for } q \neq 1.$$

In particular, if  $x_0$  is uniformly asymptotically stable, then this holds for each triangulable (+)-invariant set sufficiently near to  $x_0$ .

Corollary 3. Let  $C$  be a cycle with period  $\Theta > 0$  of a dynamical system on  $P$ . Then, for any triangulable (+)-invariant set  $X \subset P$  such that

$\lim_{n \rightarrow \infty} x \tau n\Theta \in C$ , uniformly for  $x \in X$ , there hold

$$\begin{aligned} \pi_0(X) &= \pi_1(X) = 1, \\ \pi_q(X) &= 0 \quad \text{for } 0 \neq q \neq 1. \end{aligned}$$

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