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ON PROJECTIONS IN BANACH SPACE

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The connections between linear and non-linear projections in certain types of Banach spaces are studied in this note.

Let Y be a Banach space and X a closed linear subspace of Y . By a projection from Y into X we mean in general a mapping P from Y onto X such that $Px = x$ for $x \in X$. There may be considered many natural types of projections in Banach spaces, such as linear projections, continuous projections, uniformly continuous projections, bounded linear projections, norm preserving projections, etc.

The study of projections in Banach spaces has a close relation with the study of extending mappings of X into a Banach space Z to a map of Y into Z , with the respective properties. Such a connection is shown for example in [5].

It is well known that there always exists a linear projection from Y onto X and (see [4]) a continuous projection from Y onto X . On the other hand it is known that a projection which is both linear and continuous need not exist in a Banach space (c.f., e.g. [2] pp 94-96).

The main problem studied in this paper is the existence of projections which are intermediate between continuous projections and bounded linear ones. We shall consider mostly uniformly continuous projections and projections which satisfy a Lipschitz condition.

It will be shown that in many situations the existence of a uniformly continuous projection from Y onto X (where X is a closed linear subspace of Y) implies the existence of a bounded linear projection. This is the case for example, if X is a space conjugate to a Banach space. In view of known results concerning the non-existence of bounded linear projections from Y onto X, one may thus obtain results on non-existence of uniformly continuous (nonlinear) projections.

Simultaneously with results concerning projections we shall obtain some theorems on uniformly continuous lifting problems.

All Banach spaces to be considered are taken over the real number field. Let T be a mapping from a metric space (E, ρ) into a metric space (E', σ) ; then the function

$$\varphi(\varepsilon) = \sup_{\rho(x_1, x_2) < \varepsilon} \sigma(Tx_1, Tx_2) ,$$

defined for $\varepsilon > 0$, is called the module of continuity of T ; T is uniformly continuous if and only if

$$\lim_{\varepsilon \to 0+} \varphi(\varepsilon) = 0 .$$

In the case that E is a convex subset of Banach space it can be proved that the module of continuity of every uniformly continuous mapping T is subadditive, that is

$$\varphi(\varepsilon_1 + \varepsilon_2) \leq \varphi(\varepsilon_1) + \varphi(\varepsilon_2),$$

$$\varepsilon_1 \geq 0 , \quad j = 1, 2 .$$

A special type of uniformly continuous mappings are the Lipschitzian mappings, that is mappings from (E, ρ) into

(E', σ) for which

$$\| T \| = \sup_{x_1 \neq x_2} \frac{\sigma(Tx_1, Tx_2)}{\rho(x_1, x_2)} \quad \text{is finite.}$$

The number $\| T \|$ is called a norm of Lipschitzity of the mapping T.

The following lemma is a consequence of subaddivity of the modules of continuity, and the proof is obvious.

Lemma: Let E be a convex subset of Banach space, and T a uniformly continuous mapping from E into a metric space Y. Then for each $\eta > 0$ there exists a $\lambda < +\infty$ such that

$$\rho(x_1, x_2) \geqq \eta \Rightarrow \sigma(Tx_1, Tx_2) \leqq \lambda \rho(x_1, x_2).$$

Let X be a Banach space; then X^* denotes in the usual sense the space of all continuous linear functionals in X. The symbol $X^\sim$ denotes the space of all Lipschitzian mappings from X into the real number field E_1 which vanish at the origin of X. The norm of an f in $X^\sim$ is defined as the constant of Lipschitzity of f. It is clear that $X^\sim$ is a Banach space and X^* is a closed subspace of $X^\sim$. The special case of the results of Aronszajn and Panitchpakdi [1] is an analogue to the Hahn-Banach theorem which holds for the space $X^\sim$.

As first step we shall prove that, under certain assumptions, the existence of uniformly continuous projections implies the existence of a Lipschitzian projection.

Theorem 1. Let Y be a Banach space and X a closed linear subspace of Y. If there exists a uniformly continuous projection P from Y onto X and a Lipschitzian projection Q from X^{**} onto X, then there exists a Lipschitzian projection from Y onto X .

Proof: Lemma 1 implies the existence of a number λ such that if $\| y_1 - y_2 \| \geq 1$ then

$$\| Py_1 - Py_2 \| \leq \lambda \| y_1 - y_2 \| .$$

The seqence $\{P_n\}$ of projections

$$P_n (y) = \frac{1}{n} P(n\, y)$$

may be considered as a sequence of projections from Y into X^{**} such that for every fixed $y \neq 0$ and $n > \| y \|^{-1}$ there is

$$\| P_n(y) \| \leq \lambda \| y \| ;$$

thus the sequence $P_n(y)$ is bounded at every point $y \in Y$. Since cells in X^{**} are w^*-compact, there exists a subsequence $\{P_{k_n}(y)\}$ of $\{P_n(y)\}$ with a w^*-limit $P_0(y)$ for each $y \in Y$. Because $P_n(x) = x$ for every $x \in X$, there is $P_0(x) = x$ for each $x \in X$, and the inequality

$$\| P_n\, y_1 - P_n\, y_2 \| \leq \lambda \| y_1 - y_2 \|$$

holding for $n > \| y_1 - y_2 \|^{-1}$ implies that $\| P_0 \| \leq \lambda$. A consequence of the assumption of the existence of a projection Q is the existence of the mapping $Q\, P_0$ from Y into X which is Lipschitzian.

The following theorem is the dual result for lifting operators. The proof of this is similar to that of theorem 1, and therefore we shall omit details.

Theorem 2. Let Y be a reflexive Banach space and X a quotient space (with the quotient norm) of Y . Let R be the quotient mapping. If there exists a uniformly continuous mapping T from X into Y such that R T is the identity of

X , then there is a Lipschitzian mapping T_o from X into Y such that $R\,T_o$ is the identity.

Proof: Again consider the sequence

$$T_n(x) = \frac{1}{n} T(n\,x), \qquad n = 1,2, \dots ,$$

of mappings from X into Y . For each $n = 1,2, \dots,$ $R\,T_n$ is the identity of X . Let T_o be a limit of $\{T_{k_n}\}$ in the pointwise convergence topology, taking the w-topology in Y . The mapping T_o is Lipschitzian and $R\,T_o$ is the identity of X .

Remark: I do not know whether, for every Banach space X , there exists a Lipschitzian projection from X^{**} onto X . In a large class of spaces, for example for all conjugate spaces and also for the space L_1 , there exists a linear projection from X^{**} onto X with norm 1 (cf. [3]), but this is not valid for example in the space c_o .

Theorem 3: Let Y be a Banach space and X a closed linear subspace of Y . Then there exist linear projections of norm 1 , P_Y from $Y^{\sim}$ onto Y^* and P_X from $X^{\sim}$ onto X^*, such that $P_X R_1 = R_2 P_Y$, where R_1 and R_2 are the natural restriction mappings from $Y^{\sim}$ onto $X^{\sim}$ and Y^* onto X^* respectively.

Proof: 1) First we shall construct a linear projection from $Y^{\sim}$ onto Y^* in the case that Y is of finite dimension.

Denote by ℓ_i, $i = 1, \dots,$ a basis of Y and take a function ψ on Y which is non-negative, belongs to the class $C^1(Y)$ (i.e. is continuously differentiable in each variable), has a compact support in Y and satisfies the condition

$$\int_Y \psi(y)\, dy = 1 .$$

Let $F \in Y^{\sim}$, and define

(1) $$P F \left(\sum_{i=1}^{m} \alpha_i \ell_i\right) = - \sum_{i=1}^{m} \alpha_i \int_Y F(y) \frac{\partial \psi}{\partial y_i}(y)\, dy .$$

It is clear that P is a linear mapping from $Y^{\sim}$ into Y^*.

1A) First assume that $F \in C^1(y)$. Because ψ has a compact support,

$$-\int_Y F(y) \frac{\partial \psi}{\partial y_i}(y)\, dy = \int_Y \frac{\partial F}{\partial y_i}(y)\, \psi(y)\, dy, \quad i = 1, \ldots, m .$$

Hence it is clear that $PF = F$ for $F \in Y^*$ (in particular, $\| P \| \geq 1$). Let us estimate the norm of $P F$. Assuming that $\lambda > 0$ and $\| \sum_{i=1}^{n} \alpha_i \ell_i \| = 1$,

$$P F \left(\sum_{i=1}^{n} \alpha_i \ell_i\right) = - \frac{1}{\lambda} \int_Y \psi(y) \left(\sum_{i=1}^{n} \lambda \alpha_i \frac{\partial F}{\partial y_i}(y)\right) dy =$$

$$= - \frac{1}{\lambda} \int_Y \psi(y) \left[F\left(y + \sum_{i=1}^{n} \lambda \alpha_i \ell_i\right) - F(y) + \lambda\, \vartheta(\lambda, y)\right] dy ,$$

where $\vartheta(\lambda, y) \longrightarrow 0$ uniformly with $\lambda \to 0$ in the support of ψ.

Since $\| F(y + \sum_{i=1}^{n} \lambda \alpha_i \ell_i) - F(y) \| \leq \lambda \| F \|$ and $P F (\sum_{i=1}^{n} \alpha_i \ell_i)$ does not depend on λ, we obtain the inequality

$$\left| P F \left(\sum_{i=1}^{n} \alpha_i \ell_i \right)\right| \leq \| F \| \int_Y \psi(y)\, dy = \| F \| ,$$

and hence $\| P F \| \leq \| F \|$.

1B) Now let F be a general element of $Y^{\sim}$. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of non-negative C^1 functions on Y satisfying the condition

$$\int_Y \varphi_n(y)\, dy = 1$$

and such that the support of φ_n is in the cell $\| y \| \leq \frac{1}{n}$.

Let $*$ denote the convolution of functions. Set

$$F_n = F * \varphi_n \ .$$

Then $F_n(y) \to F(y)$ uniformly on Y as $n \to \infty$, $F_n \in Y^{\sim} \cap C^1$ and $\| F_n \| \leq \| F \|$. From (1) it follows that

$$\| P F_n - P F \| \to 0$$

and the inequality $\| P F_n \| \leq \| F_n \|$ implies that $\| P F \| \leq \leq \| F \|$ and hence $\| P \| = 1$.

If now X is a linear subspace of the finite dimensional space Y, we may assume that the basis of Y has the property that $\ell_1, \dots, \ell_k$ is the basis of X for some $k \leq m$. Let ψ_1, ψ_2 be functions of k and $m-k$ variables respectively, having compact supports and such that

$$\int_X \psi_1(x)\, dx = \int_{Y/X} \psi_2(z)\, dz = 1,$$

where $x = (y_1, \dots, y_k)$, $z = (y_{k+1}, \dots, y_m)$.

For every positive integer n let P_n be the linear projection from $Y^{\sim}$ onto Y^* of norm 1, defined for $F \in Y^{\sim}$ by the analogue of (1),

$$P_n F(\ell_i) = - n^{k-m} \int_Y F(y) \frac{\partial}{\partial y_i} [\psi_1(y_1 \dots, y_k) \cdot$$

$$\cdot \psi_2(n y_{k+1}, \dots, n y_m)]\, dy \ .$$

Since $\| P_n \| \leq 1$ and Y^* is of finite dimension, the unit cell of $(Y^{\sim})^*$ is w^*-compact and hence the subsequence $\{P_{k_n}\}_{n=1}^{\infty}$ of linear operators from $X^{\sim}$ to Y^* has a limit P_Y in the strong operator topology. The mapping P_Y is a linear projection of norm 1 from $Y^{\sim}$ onto Y^*, and uniform continuity of F implies that

$$P_Y F(\ell_i) = \lim_{n\to\infty} P_{k_n} F(\ell_i) = - \int_X F(y_1, \dots, y_k, 0, \dots 0) \cdot$$

$$\cdot \frac{\partial \psi_1}{\partial y_1} (y_1, \dots, y_k)\, dx .$$

Define a mapping P_X from $X^\sim$ onto X^* for $G \in X^*$ by

$$P_X G(\ell_i) = - \int_X G(x) \frac{\partial \psi_1}{\partial x_i} (x)\, dx, \qquad i = 1, \dots, k .$$

If R_1 and R_2 are the restriction operators from $Y^\sim$ onto $X^\sim$ and from Y^* onto X^* respectively, we thus have

$$R_2 P_Y = P_X R_1 ,$$

and the proof for the case that Y is of finite dimension is concluded.

2) Let now Y be an infinite dimensional Banach space and X a closed linear subspace of Y. Let B be a finite dimensional subspace of Y and $C = B \cap X$. Let $R_{1,B}$ and $R_{2,B}$ be the restriction mappings from $B^\sim$ onto $C^\sim$ and B^* onto C^* respectively. In view of the first part of the proof there exist linear projections of norm 1, P_B from $B^\sim$ onto B^* and P_C^B from $C^\sim$ onto C^* respectively, such that

$$R_{2,B} P_B = P_C^B R_{1,B} .$$

Next, let R_1^B and R_1^C be the restriction mappings from $Y^\sim$ onto $B^\sim$ and $X^\sim$ onto $C^\sim$ rexpectively. For $F \in Y^\sim$ let

$$\mathcal{F}_B (F, y) = \begin{cases} 0 & \text{if } y \notin B \\ P_B R_1^B F(y) & \text{if } y \in B ; \end{cases}$$

this maps $Y^\sim \times Y$ into E_1; and for $G \in X^\sim$ let

$$\mathcal{G}_B (G, x) = \begin{cases} 0 \text{ if } x \in B \\ P_C^B R_1^C G(x) & \text{if } x \in B ; \end{cases}$$

this maps from $X \times X$ into E_1.

For every $F \in Y^{\sim}$, $y \in Y$, $G \in X^{\sim}$, $x \in X$ and each B there is

$|\mathcal{F}_B(F, y)| \leqq \|F\| \|y\|$ and $|\mathcal{G}_B(G, x)| \leqq \|G\| \|x\|$.

Consider the directed system Φ of all finite dimensional subspaces of Y ordered by inclusion, and the corresponding nets of functions

$$\{\mathcal{F}_B\}_{B \in \Phi} \quad \text{and} \quad \{\mathcal{G}_B\}_{B \in \Phi}$$

By the Tichonov theorem there exist subnets

$$\{\mathcal{F}_{B'}\}_{B' \in \Phi'} \quad \text{and} \quad \{\mathcal{G}_{B'}\}_{B' \in \Phi'}$$

which are pointwise convergent, say to mappings f from $Y^{\sim} \times Y$ into E_1, and g from $X^{\sim} \times X$ into E_1, respectively.

The mapping $\mathcal{F}_B(F, y)$ is linear in F for each B, and

$$\mathcal{F}_B(F, \alpha y_1 + \beta y_2) = \alpha \mathcal{F}_B(F, y_1) + \beta \mathcal{F}_B(F, y_2)$$

for every B which contains both y_1 and y_2. Hence

$$f(F(y)) = P_Y F(y),$$

where P_Y is a linear operator from $Y^{\sim}$ into Y, with norm $\|P_Y\| \leqq 1$.

Because $\mathcal{F}_B(F, y) = F(y)$ for $F \in Y^*$ and $y \in B$ we have, for all $y \in Y$ and $F \in Y^*$, that

$$f(F, y) = F(y).$$

Therefore P_Y is a projection from $Y^{\sim}$ onto Y^*.

Applying this result to X in place of Y one obtains that there exists a projection P_X from $X^{\sim}$ onto X^* such that

$$g(G, x) = P_X G(x).$$

For every $F \in Y^{\sim}$ and $x \in X \cap B = C$ we have

$$\mathcal{F}_B(F, x) = R_{2,B} P_B R_1^B F(x) = P_C^B R_{1,B} R_1^B F(x) =$$

$$= P_C^B R_{1,C} R_1 F(x) = \mathcal{G}_B(R_1 F, x) ,$$

and because $R_{1,B} R_1^B = R_{1C} R_1$, these mappings are both restriction mappings from $Y^\sim$ onto $C^\sim$.

Hence

$R_2 P_Y F(x) = f(F, x) = g(R_1 F, x) = P_X R_1 F(x)$

for $F \in Y^\sim$ and $x \in X$. This concludes the proof of the theorem.

Theorem 4. Let X be a closed linear subspace of a Banach space Y . If there exists a Lipschitzian projection with norm λ from Y into X , then there exists a linear operator T with norm $\| T \| \leq \lambda$ from X^* into Y^* such that, if R_2 is the restriction mapping from Y^* onto X^* , then $R_2 T$ is the identity of X^* .

Proof: Use the notation of the proof of theorem 3 . Let P_Y and P_X satisfy

$$R_2 P_Y = P_X R_1 .$$

Let P be a Lipschitzian projection from Y onto X with norm $\| P \| = \lambda$, and let $P^\sim$ be the linear mapping from $X^\sim$ into $Y^\sim$ defined by

$$P^\sim G(y) = G(P y) \quad \text{for } G \in X^\sim , y \in Y .$$

Clearly $\| P^\sim \| \leq \lambda$, and $R_1 P^\sim$ is the identity of $X^\sim$.

Set

$$T = P_Y P^\sim ;$$

T is a linear mapping from $X^\sim$ into Y^* with norm $\| T \| \leq \lambda$.

For $x^* \in X^*$ we have

$$R_2 T x^* = R_2 P_Y P^\sim x^* = P_X R_1 P^\sim x^* = P_X x^* = x^*$$

and hence the restriction of T to X^* is the required mapping.

The dual lifting result is

Theorem 5. Let X be a quotient space of a Banach space Y, and R the corresponding quotient mapping. If there is a Lipschitzian mapping T of norm $\| T \| \leq \lambda$ from X into Y such that $R\,T$ is the identity of X, then there exists a linear projection of norm at most λ from Y^* onto X^*.

Proof: Let $T' = T - T(0)$, therefore $T'(0) = 0$. Define $T'^{\sim}$ from $Y^{\sim}$ onto $X^{\sim}$ by

$$T'^{\sim} F(x) = F(T'x) \quad \text{for} \quad x \in X, \quad F \in Y^{\sim} ;$$

$T'^{\sim}$ is a linear projection of norm $\| T'^{\sim} \| \leq \lambda$. Let us identify an element $G \in X^{\sim}$ with the element $\hat{G} \in Y^{\sim}$ defined by

$$\hat{G}\,(y) = G\,(R\,y) ,$$

and the element $x^* \in X^*$ with the element $\hat{x}^* \in Y^*$ defined by

$$\hat{x}^*(y) = x^*(R\,y) .$$

Let P_X be a linear projection of norm 1 from $X^{\sim}$ onto X^*. Then the restriction of $P_X T'^{\sim}$ to Y^* is a linear projection with norm at most λ from Y^* onto X^*.

Now we shall develop some easy corollarys of foregoing theorems.

Corollary 1: Let Y be a Banach space and X a subspace of Y. If there exists a Lipschitzian projection of norm λ from Y onto X, then there is a linear projection of the norm at most λ from Y^{**} onto X^{**}.

Proof: Let T be a mapping from X^* into Y^* whose existence is shown in Theorem 4. T^* is the required linear projection.

I am unable to assert whether the assumption of corollary 1 is sufficient for the existence of a bounded linear projection from Y onto X. It may be proved with the following supplementary assumptions:

Corollary 2: Let Y be a Banach space and X a closed linear subspace of Y. If there exists a bounded linear projection from X^{**} onto X, and a uniformly continuous projection from Y onto X, then there exists a bounded linear projection from Y onto X.

Proof: By Theorem 1 and the Corollary 1, there exists a bounded linear projection from Y^{**} onto X^{**}.
If Q is a bounded linear projection from X^{**} onto X it is easy to show that the restriction of $Q\ P$ to X has the required properties.

Corollary 3: Let Y be a reflexive Banach space, X a quotient Banach space of Y and R the quotient mapping. If there exists a uniformly continuous mapping from X into Y such that $R\ T$ is the identity of X, then there also exists a bounded linear mapping T_0 from X into Y such that $R\ T_0$ is the identity of X.

This proposition is an easy consequence of theorems 1 and 5.

R e f e r e n c e s :

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