Aleš Pultr

Concerning universal categories

Commentationes Mathematicae Universitatis Carolinae, Vol. 5 (1964), No. 4, 227--239

Persistent URL: http://dml.cz/dmlcz/104979

Commentationes Mathematicae Universitatis Carolinae
5, 4 (1964)

CONCERNING UNIVERSAL CATEGORIES

A. PULTR, Praha

Introduction.

The present paper contains a definition of universal category and some results concerning this notion. In the § 1 we give the definition and two simple criteria which enables us to show that some frequently discussed categories are not universal. At the end of the paragraph we prove that the category of commutative semi-groups and their homomorphisms is not universal, which result is not an immediate consequence of the criteria.

The category $\mathcal{R}$ of sets with binary relations and their compatible mapping is shown to be universal in the § 2. A proof of the fact was sketched in [2] using the constructions introduced in [1]. Here it is done in a more detailed way.

The main results of the paper are given in the § 3. We examine some categories similar to $\mathcal{R}$ and state which of them are universal. Every transformation $\varphi : X \rightarrow X$ may be considered as a particular case of a relation on X, namely $\{(\varphi(x), x) \mid x \in X\}$. Another particular kind of relations form multivalued mappings $\mu : X \rightarrow X$, i.e. such relations, that for every $x \in X$ there is at least one y with $y \mu x$. Roughly speaking, the definition of $\mathcal{R}$ contains two kinds of relations: namely, relations in objects and morphisms, which are, in fact, a particular kind of compatible generalized relations. To suggest it, we write $\mathcal{R} = \mathcal{R}(r, f)$. We may replace r by f, m or r and the same for f (r means relations, m multivalued mappings, f mappings). In this way we obtain nine categories $\mathcal{R}(\alpha, \beta)$. Considering another kind of compatibility, we shall define another nine categories $\mathcal{R}^*(\alpha, \beta)$. Exact definitions of the categories $\mathcal{R}(\alpha, \beta)$, $\mathcal{R}^*(\alpha, \beta)$ are given in the beginning of § 3.

We remark that throughout the § 3 we work in a set theory without inaccessible cardinals.

§ 1.

M(a, b) denotes, as usual, the set of all the morphisms from an object a to an object b (in a given category). A class obtained by choicing a representant in every equivalence class (given by isomorphism) of A, where A is a class of objects, is called a skeleton of A. Small category is a category such that the class of all its objects is a set.

1.1. Definition. A category $\mathcal{K}$ is called universal, if any small category is isomorphic to a full subcategory of $\mathcal{K}$.

The following statement is evident:

1.2. Theorem. The dual category of a universal category is universal. If $\mathcal{K}$ is isomorphic to a full subcategory of $\mathcal{L}$, and if $\mathcal{K}$ is universal, then $\mathcal{L}$ is universal.

1.3. Remark. Since semigroups with unity elements (in particular, groups) are small categories, the following holds: Let $\mathcal{K}$ be a universal category, S^1 any semigroup with unity element. Then there exists an object a in $\mathcal{K}$ such that S^1 is isomorphic to M(a, a).

1.4. Theorem. Let $\mathcal{K}$ have the following property: There exists a class A of objects of $\mathcal{K}$ such that its skeleton is a set and such that for every $a \in \mathcal{K} \setminus A$ there exists a $b \in \mathcal{K}$ with $M(a, b) \neq \emptyset \neq M(b, a)$ and such that a is not a direct summand of b [x]. Then the category $\mathcal{K}$ is not universal.

Proof. Let the category $\mathcal{K}$ be universal and let G be any group; let us denote $a(G)$ an object of $\mathcal{K}$ such that $G \approx M(a(G), a(G))$. Obviously, if G, G' are not isomorphic, the objects $a(G), a(G')$ are not isomorphic. If $a = a(G) \notin A$, there exist $b, \varphi, \psi \in \mathcal{K}$, $\varphi : a \to b$, $\psi : b \to a$, a being not a direct summand of b. Therefore $\psi \circ \varphi$ has not an inverse morphism and we have a contradiction, for $M(a, a)$ is a group. Hence $a(G) \in A$ for any group. But there is a proper class of mutually non-isomorphic groups.

1.5. Theorem. Let $\mathcal{K}$ have the following property: There exists a class A of objects of $\mathcal{K}$ such that its skeleton is a set and such that for every $b \in \mathcal{K}$ there exists an $a \in A$ such that $M(a, b) \neq \emptyset \neq M(b, a)$. Then the category $\mathcal{K}$ is not universal.

Proof. We are going to obtain the statement by proving that any object of any category has only a set of mutually non-isomorphic direct summands. Let us have an object a. Let b, b' be its non-isomorphic direct summands. Let $\alpha : b \to a$, $\beta : a \to b$, $\alpha' : b' \to a$, $\beta' : a \to b'$ be morphisms such that $\beta \circ \alpha$, $\beta' \circ \alpha'$ are isomorphisms. $\alpha \circ \beta = \alpha' \circ \beta'$ implies $(\beta \circ \alpha') \circ$ $\circ (\beta' \circ \alpha) = \beta \circ \alpha \circ \beta \circ \alpha$ and $(\beta' \circ \alpha) \circ (\beta \circ \alpha') =$ $= \beta' \circ \alpha' \circ \beta' \circ \alpha'$, i.e. b isomorphic with b', in a contradiction with the assumption. The assertion concerning the set of mutually non-isomorphic direct summands is therefore an immediate consequence of the fact that $M(a, a)$ is a set.

----

x) a is called a direct summand of b, if there exist morphisms $\alpha : a \to b$, $\beta : b \to a$ such that $\beta \circ \alpha$ is an isomorphism.

1.6. Corollary. If there is an object $a \in \mathcal{K}$ such that both $M(b, a)$ and $M(a, b)$ are non-void for any $b \in \mathcal{K}$, in particular, if $\mathcal{K}$ has a singleton resp. cosingleton, then $\mathcal{K}$ is not a universal category.

1.7. The criteria 1.4 and 1.5 do not give an immediate answer to the question whether the category of semigroups (in general, without a unity element) is universal. We close this paragraph by showing that at least the category of commutative semigroups and their homomorphisms is not universal. The statement is an immediate consequence of the following theorem:

Theorem. Let S be a semigroup, $a \in S$ an element such that $a^m = a^m$ for some $m \neq m$. Then there is an element $b \in S$ with $b \cdot b = b$. Consequently, a commutative semigroup either contains an element b with $b \cdot b = b$, or its endomorphism semigroup is infinite (more precisely, it contains an isomorphic image of the semigroup of natural numbers with multiplication).

Proof. Let $a^m = a^m$ with $n > m$. Hence $a^{n-m} \cdot a^m = a^m$ and consequently $a^m = a^{k(n-m)} \cdot a^m$. Let us take a k such that $k(n - m) \quad m$. Multiplying the both sides of the last equation by $a^{k(n-m)-m}$ we get $b \cdot b = b$ for $b = a^{k(n-m)}$. Now, let S be a commutative semigroup; let $b \cdot b \neq b$ for every $b \in S$. Hence for any a and any $m \neq n$ holds $a^m \neq a^n$. Let us denote N the semigroup of natural numbers. The mapping $\Phi : N \to E(S)$ defined by $\Phi(n)(x) = x^m$ is evidently a monomorphism.

§ 2.

2.1. Denotations. Let X, Y be sets, R, S binary relations on X, Y respectively. By a compatible (more precisely, RS-compatible) mapping $f : (X, R) \to (Y, S)$ we mean a mapping $f : X \to Y$ such that the implication $x R x' \Rightarrow f(x) S f(x')$

holds. The sets with binary relations and their compatible mappings form obviously a category, which will be denoted by $\mathcal{R}$. Let us denote $\mathcal{R}_a$ ($\mathcal{R}'_a$ resp.) the full subcategory of $\mathcal{R}$ generated by objects (X, R) with antireflexive R such that for every $x \in X$ there is a $y \in X$ with $y R x$ (with either $x R y$ or $y R x$).

Let A be a set. We denote by $A\mathcal{R}$ the category, described as follows: objects are systems $(X ; \{R_a\} , a \in A)$ where X is a set and every R_a is a binary relation on X , and the morphisms from $(X ; \{R_a\} , a \in A)$ into $(Y ; \{S_a\}, a \in A)$ are all the mappings $f : X \to Y$, which are $R_a S_a$ -compatible for every $a \in A$. Let (X, R) be a set with a binary relation. We denote $C(X, R)$ the semigroup of all the compatible mappings of (X, R) into itself. The object (X, R) is said to be rigid, if $C(X, R)$ is trivial.

2.2. Theorem. Any small category $\mathfrak{K}$ is isomorphic to a full subcategory of some $A\mathcal{R}$. The set $\mathfrak{K}'$ of the morphisms of $\mathfrak{K}$ may be taken as the set A .

Proof. Let us denote by K the set of the objects of $\mathfrak{K}$. Let us define $\Phi(a) = (\bigcup\{M(b,a) \mid b \in K\} ; \{R_\alpha\}, \alpha \in \mathfrak{K}')$, where $\beta R_\alpha \gamma$ iff $\beta = \gamma \circ \alpha$; $\Phi(\beta) = \{\gamma \to \beta \circ \gamma\}$. Obviously, Φ is a 1-1 functor into $\mathfrak{K}'\mathcal{R}$. Let $f : \Phi(a) \to \Phi(b)$ be a morphism in $\mathfrak{K}'\mathcal{R}$. Let us denote ε the identity morphism of b . Since $\alpha = \varepsilon \circ \alpha$, we have $\alpha R_\alpha \varepsilon$ and hence $f(\alpha) R_\alpha f(\varepsilon)$, i.e. $f(\alpha) = f(\varepsilon) \circ \alpha$. Hence $f = \Phi(f(\varepsilon))$.

2.3. Lemma. Let $(X, R) \in \mathcal{R}'_a$. Then there exists a $(Y, S) \in \mathcal{R}_a$ with $\operatorname{card} Y \geq \operatorname{card} X$ and $C(X, R) \approx C(Y, S)$, such that the length of any cycle x) in (Y, S) is divisible by either

x) Let $(X,R) \in \mathcal{R}$. A sequence $x_1, x_2, \dots, x_n$ $(x_i \in X)$ with the property $x_i R x_{i+1}$ $(i=1,\dots,n-1)$, $x_n R x_1$ is called a cycle of the length n.

2 or 3 .

Proof. Let us take $Y = X \cup R \cup U$, where $U = U^1 \cup U^2$, $U^1 = \{u_1^1, u_2^1\}$, $U^2 = \{u_1^2, u_2^2, u_3^2\}$. Let us define the relation S by:

$u_1^1 \, S \, u_2^1$, $u_2^1 \, S \, u_1^1$, $u_i^2 \, S \, u_{i+1}^2$ $(i = 1, 2)$, $u_3^2 \, S \, u_1^2$;
for every $x \in X$, $i = 1,2$, $u_1^i \, S \, x$;
for every $(x, y) \in R$ $x \, S(x, y)$, $(x, y) \, S \, y$.

Evidently, the length of any cycle in (Y, S) is divisible by either 2 or 3 . Since the image of any cycle under a compatible mapping is a cycle of the same length, we get immediately $\varphi(U^i) \subset U^i$ $(i = 1,2)$ for any compatible $\varphi \in C(Y, S)$. Now, let us take an $x \in X$. Since $u_1^i \, S \, x$, we have $\varphi u_1^i \, S \, \varphi x$ and we easily obtain $\varphi u_j^i = u_j^i$ and $\varphi(X) \subset X$. Let us take $(x, y) \in R$. Since $x \, S(x, y)$ and $(x, y) \, S \, y$, we have $\varphi x \, S \, \varphi(x, y)$ and $\varphi(x, y) \, S \, \varphi y$ and we get $\varphi(x, y) = (\varphi x, \varphi y)$ by the fact that $\varphi(X) \subset X$.

Now, it is easy to see that the mapping $\Phi : C(Y, S) \to C(X,R)$ defined by $\Phi(\varphi)(x) = \varphi x$ is an isomorphism.

2.4. Corollary. Let $\mathfrak{m}$ be a cardinal less than the first inaccessible one. Then there is a rigid object (X, R) in $\mathcal{R}_a$ with card $X \geq \mathfrak{m}$, such that the length of any cycle in (X, R) is divisible by either 2 or 3 .

Proof. By [1] there exists a rigid (X', R') in $\mathcal{R}$ with card $X \geq \mathfrak{m}$. Without a loss of generality we may suppose $\mathfrak{m} > 1$. Then $(X', R') \in \mathcal{R}'_a$, since if $x_0 R' x_0$ the constant mapping into x_0 is compatible, and, further, a point which is not in the relation with any other may be compatibly mapped everywhere. Now, we get the statement using 2.3 .

2.5. Theorem. In a set theory without inaccessible cardinals the category $\mathcal{R}_a$ (and hence $\mathcal{R}$, too) is universal.

Proof. By 2.2 it is sufficient to prove that, for card A accessible, the category $A\mathcal{R}$ is isomorphic to a full subcategory of $\mathcal{R}_a$. Let us take a rigid object $(B, S) \in \mathcal{R}_a$ with card B $\geq$ card A + 1 with cycles of length divisible by either 2 or 3 . Let $p_1, \ldots, p_4$ be mutually different primes, $p_i \neq 2, 3$, and let U_i $(i = 1, \ldots, 4)$ consist of formal elements $u_i(1), \ldots, u_i(p_i)$; let, finally, $U = U_1 \cup \ldots \cup U_4$.

Since card A + 1 $\leq$ card B we may choose a 1-1 mapping $\alpha : A \to B$ such that there is a $b_0 \in B \setminus \alpha(A)$. Let us define $B_i = \{(a, i) \mid a \in B\}$, $S_i = \{((a, i), (a', i)) \mid (a, a') \in S\}$ $(i = 1,2)$. Let us define $\alpha_i : A \to B_i$ by $\alpha_i a = (\alpha a, i)$. Let us, for an object $\hat{X} = (X ; \{R_a\} , a \in A) \in A\mathcal{R}$ denote $Y_a = \{(x, y, a) \mid (x, y) \in R_a\}$, $Y = \bigcup\{Y_a \mid a \in A\}$ and define $\Phi(\hat{X}) = (X_1, R_1)$, where $X_1 = X \cup Y \cup U \cup B_1 \cup B_2$ and R_1 is defined as follows:

$$u_i(j)\, R_1\, u_i(j + 1) \quad (i = 1, \ldots, 4 ,\ j = 1,\ldots, p_i - 1) ,$$

$$u_i(p_i)\, R_1\, u_i(1) \quad (i = 1,\ldots, 4) ;$$

for every $(b, 1) \in B_1$, $u_i(1)\, R_1\, (b, 1)$ $(i = 1,2)$;

for every $(b, 2) \in B_2$, $u_i(1)\, R_1\, (b, 2)$ $(i = 3,4)$;

$$(b, i)\, R_1\, (b', i) \text{ iff } b\, S\, b' \quad (i = 1,2) ;$$

for every $x \in X$, $(b_0, i)\, R_1\, x$ $(i = 1,2)$;

for every $a \in A$, $(x, y) \in R_a$, $x\, R_1 (x,y,a)\, R_1\, y$, $(x,y,a) R_1 y$, $\alpha_i a\, R_1 (x,y,a)$ $(i = 1,2)$.

Now, let $\varphi : (X; \{R_a\} , a \in A) \to (X'; \{R'_a\} , a \in A)$ be a morphism. $\Phi(\varphi) : \Phi(\hat{X}) \to \Phi(\hat{X}')$ is defined as follows:

$$\text{for } x \in X \quad \Phi(\varphi) x = \varphi x , \quad \Phi(\varphi)(x,y,a) = (\varphi x, \varphi y, a) ,$$

$$\Phi(\varphi) \text{ identical over } U \cup B_1 \cup B_2 .$$

Obviously, Φ forms a 1-1 functor into $\mathcal{R}_a$. Now, let $f : \Phi(\hat{X}) \to \Phi(\hat{X}')$ be a morphism. Similarly as in 2.3 we get $f(u_i(j)) = w_i(j)$, $f(B_i) \subset B_i$. By the rigidity of (B, S) immediately $f(b, i) = (b, i)$ for every $b \in B$. Since, particularly, $f(b_0, i) = (b_0, i)$, $f(\alpha_i a) = \alpha_i a$, we get $f(X) \subset X'$ and $f(Y_a) \subset Y'_a$. By $x R_1 (x, y, a)$, $(x, y, a) R_1 y$ we have $fx R'_1 f(x, y, a)$, $f(x, y, a) R'_1 f y$ and hence $f(x, y, a) = (fx, fy, a)$. Hence, finally, $f = \Phi(\varphi)$, where $\varphi : \hat{X} \to \hat{X}'$ is defined by $\varphi x = f x$.

§ 3.

3.1. Denotations. Let X, Y, Z be sets, $A \subset Z \times Y$, $B \subset Y \times X$. The set $\{(z, x) \mid$ there exists a $y \in Y$, $(z, y) \in A$, $(y, x) \in B\}$ is denoted by $A \circ B$. The category $\mathcal{R}(r, r)$ ($\mathcal{R}^*(r, r)$ respectively) is defined as follows: Its objects are couples (X, R), where X is a set, $R \subset X \times X$, the morphisms from (X, R) into (X', R') are triplets $(S, (X, R), (X', R'))$ such that $S \subset X' \times X$ and $S \circ R \subset R' \circ S$ ($S \circ R = R' \circ S$ respectively). The composition of morphisms is defined by the formula

$$(S', (X', R'), (X'', R'')) \circ (S, (X, R), (X', R')) = (S' \circ S, (X, R), (X'', R'')) .$$

We associate with every object (X, R) the adjoint morphism $\mathcal{a}(X, R) = (R, (X, R), (X, R))$.

A multivalued compatible mapping (strongly compatible mapping, respectively) is a morphism $(S, (X, R), (X', R'))$ such that for every $x \in X$ there exists a x' with $(x', x) \in S$. If there is always exactly one such x', we call the morphism a compatible (strongly compatible, resp.) mapping. If there is no danger of misunderstanding, we omit the words (strongly) compatible. Sometimes, we shall write simply S instead of $(S, (X, R), (X', R'))$.

By $\mathcal{R}(m, r)$ ($\mathcal{R}(f, r)$ resp.) is denoted the full subcategory of $\mathcal{R}(r, r)$ generated by the objects (X, R) such that $\mathcal{a}(X, R)$ is a multivalued mapping (a mapping, resp.).

By $\mathcal{R}(r, m)$ ($\mathcal{R}(r, f)$ resp.) is denoted the subcategory of $\mathcal{R}(r, r)$ consisting of the same objects as $\mathcal{R}(r, r)$ and of all their multivalued mappings (mappings, resp.). (It is easy to see that $\mathcal{R}(r, f)$ is isomorphic to the category from the § 2 .)

$\mathcal{R}(m, m)$ ($\mathcal{R}(f, m)$ resp.) is the full subcategory of $\mathcal{R}(r, m)$ generated by the objects (X, R) such that $\mathcal{a}(X, R)$ is a multivalued mapping (a mapping, resp.).

Finally, $\mathcal{R}(m, f)$ ($\mathcal{R}(f, f)$, resp.) is the full subcategory of $\mathcal{R}(r, f)$ generated by the objects (X, R) such that $\mathcal{a}(X, R)$ is a multivalued mapping (a mapping, resp.).

The categories $\mathcal{R}^*(\alpha, \beta)$ ($\alpha, \beta = r, m, f$) are defined analoguously.

The category $\mathcal{R}_a$ is evidently isomorphic to a full subcategory of $\mathcal{R}(m, f)$. Hence the categories $\mathcal{R}(m, f)$ and $\mathcal{R}(r, f)$ are universal ones. In present paragraph we shall prove that with exception of these two and $\mathcal{R}^*(m, f)$ and $\mathcal{R}^*(r, f)$ no category defined above is universal. On the other hand, we shall prove that both $\mathcal{R}^*(m, f)$ and $\mathcal{R}^*(r, f)$ are universal. Hence, the situation for both $\mathcal{R}(\alpha, \beta)$ and $\mathcal{R}^*(\alpha, \beta)$ ($\alpha, \beta = r, m, f$) is described by the following table (+ means: the category is universal, - : the category is not universal):

$\alpha \backslash \beta$	f	m	r
f	-	-	-
m	+	-	-
r	+	-	-

3.4. <u>Theorem</u>. The categories $\mathcal{R}^*(\alpha, \alpha)$ ($\alpha = r, m, f$) are not universal.

Proof. Let us consider the group P_3 of the permutations of a three-point set. If $\mathcal{R}^*(r, r)$ ($\mathcal{R}^*(m, m)$, $\mathcal{R}^*(f, f)$ resp.) were universal, there ought to be a relation R (multivalued mapping μ, mapping φ, resp.) on a set X such that P_3 were isomorphic to the semigroup of all the morphisms of (X, R) ((X, μ), (X, φ) resp.) into itself, i.e. to the semigroup of the sets $R' \subset X \times X$ (multivalued mappings $\mu' : X \to X$, mappings $\varphi' : X \to X$, resp.) with $R' \circ R = R \circ R'$ ($\mu' \circ \mu = \mu \circ \mu'$, $\varphi' \circ \varphi = \varphi \circ \varphi'$, resp.). Since R ($\mu$, φ, resp.) itself is an element of the semigroup, it has to correspond to an element of P_3, which commutes with any other one. But only the unity element possesses in P_3 this property, and, on the other hand, the unity element obviously corresponds to the diagonal Δ of $X \times X$. We got a contradiction, since, except of one-point X, the semigroup of all the morphisms from (X, Δ) into (X, Δ) is not a group.

3.3. Corollary. $\mathcal{R}^*(m, r)$, $\mathcal{R}^*(f, r)$ and $\mathcal{R}^*(f, m)$ are not universal.

3.4. Lemma. Let G be a group provided by a (reflexive) partial ordering $\prec$ such that the following implication holds:

$$x, y, z \in G, \ x \prec y \Rightarrow zx \prec zy, \ xz \prec yz .$$

Let for some $g \in G$ and for every $x \in G$ $xg \prec gx$. Then $xg = gx$ for every $x \in G$.

Proof. Let $x \in G$. Since $x^{-1}g \prec gx^{-1}$, we get $gx \prec xg$ (multiplying by x from both the right and the left), and hence, assuming $xg \prec gx$, $gx = xg$.

3.5. Theorem. $\mathcal{R}(\alpha, \alpha)$ ($\alpha = r, m, f$) are not universal.

Proof. Let the semigroup of morphisms be ordered by inclusion.

Using the lemma 3.4 we may now repeat the proof of 3.2 .

3.6. Corollary. The categories $\mathcal{R}(m, r)$, $\mathcal{R}(f, r)$ and $\mathcal{R}(f, m)$ are not universal.

3.7. Theorem. $\mathcal{R}(r, m)$ and $\mathcal{R}^*(r, m)$ are not universal.

Proof. We prove, that any non-trivial group of all the morphisms of some (X, R) into itself contains a non-trivial element, commuting with every other one. Really, $(R \cup \Delta, (X, R), (X, R))$ is a multivalued mapping. If $(S, (X, R), (X, R))$ is another one, we have

$$S \circ (R \cup \Delta) = S \circ R \cup S \circ \Delta \subset (= \text{resp.})\ R \circ S \cup \Delta \circ S = (R \cup \Delta) \circ S .$$

3.8. Theorem. The category $\mathcal{R}_a$ is isomorphic to a full subcategory of $\mathcal{R}^*(m, f)$.

Proof. Let $(X, R) \in \mathcal{R}_a$. Let us denote by X_i $(i = 1, 2)$ the set $\{(x, y, i) \mid (x, y) \in R\}$, by X_3 the set $\{(x, 3) \mid x \in X\}$. Let us define a relation $\bar{R}$ on the set $\bar{X} = X \cup X_1 \cup X_2 \cup X_3$ as follows:

for every $x \in X$ $x \bar{R}(x, 3)$ and $(x, 3) \bar{R} x$,

for every $(x, y) \in R$ $x \bar{R}(x, y, 2)$, $y \bar{R}(x, y, 2)$, $y \bar{R}(x, y, 1)$ and $(x, y,) \bar{R} (x, y, 2)$.

Let us denote $\Phi(X, R) = (\bar{X}, \bar{R})$. Let $\varphi : (X, R) \to (Y, S)$ be a compatible mapping. The mapping $\Phi(\varphi) : \Phi(X, R) \to \Phi(Y, S)$ is defined as follows:

for $x \in X$ $\Phi(\varphi)x = \varphi x$, $\Phi(\varphi)(x, 3) = (\varphi x, 3)$,

for $(x, y) \in R$, $i = 1,2$ $\Phi(\varphi)(x, y, i) = (\varphi x, \varphi y, i)$.

Evidently $\Phi(\varphi) \circ \bar{R} \subset \bar{S} \circ \Phi(\varphi)$. We are going to prove the converse inclusion. Let $(a, b) \in \bar{S} \circ \Phi(\varphi)$. First, let $b = (x, y, 2)$; we have $a \bar{S} (\varphi x, \varphi y, 2)$ and therefore the element a must be equal to either $(\varphi x, \varphi y, 1)$ or φx or

φy . In any of these cases $(a, b) \in \Phi(\varphi) \circ \overline{R}$. Let $b = (x, y, 1)$; then we have $a \overline{S} (\varphi x, \varphi y, 1)$ and hence $a = \varphi y$, so that $(a, b) \in \Phi(\varphi) \circ \overline{R}$, too. Similarly, in the case of $b = x$ ($b = (x, 3)$ resp.), which leads to $a = (\varphi x, 3)$ ($a = \varphi x$ resp.). Hence, finally, $\Phi(\varphi) \circ \overline{R} = \overline{S} \circ \Phi(\varphi)$, i.e. $\Phi(\varphi)$ is a strongly compatible mapping, and Φ is an (evidently 1-1) functor into $\mathcal{R}^*(m, f)$. It remains to prove that the image of Φ is a full subcategory of $\mathcal{R}^*(m, f)$. Let $f : (\overline{X}, \overline{R}) \to (\overline{Y}, \overline{S})$ be a strongly compatible mapping. Since there are no cycles in $(\overline{X}, \overline{R})$ and $(\overline{Y}, \overline{S})$ but cycles of a type either $x, (x, 3), x, \ldots, (x, 3)$ or $(x, 3), x, (x, 3), \ldots, x$, we have $f(\{x, (x, 3)\}) \subset \{x'', (x'', 3)\}$ for every $x \in X$. Since, for every $x \in X$, there is a $y \in X$ with $y R x$, the equality $fx = (x'', 3)$ leads (by $x \overline{R} (y, x, 1)$ and $x \overline{R}(y, x, 2)$) to the equalities $f(y, x, 1) = f(y, x, 2) = x''$, in a contradiction to $f(y, x, 1) \overline{S} f(y, x, 2)$.

Hence we have $fx = x''$, $f(x, 3) = (x'', 3)$.
Now, let us turn our attention to $f(x, y, 2)$. If $f(x, y, 2)$ is equal to either x' or $(x', 3)$ or $(x', y', 1)$, we get $f(x, y, 1) = f y$, what is not possible. We have hence $f(x, y, 2) = (x', y', 2)$ and $f(x, y, 1)$ is equal to either $(x', y', 1)$ or x' or y' . However, the second and the third case implies $fy = (x', 3)$ ($= (y', 3)$ resp.) in a contradiction with $f y \in Y$ proved above. Hence $f(x, y, 1) = (x', y', 1)$ and, consequently, $f y = y'$, and $f x$ is either the x' or the y' . But $f x = y'$ implies $\overline{S} \circ f(x, y, 2) = \overline{S}(x', y', 2) = \{(x', y', 1), x', y'\}$, and $f \circ \overline{R} (x, y, 2) = f(\{(x, y, 1), x, y\}) = \{(x', y', 1), y'\}$ only. We get $f x = x'$ and we see that $f = \Phi(\varphi)$, where $\varphi : X \to Y$ is defined by $\varphi x = f x$.

3.9. <u>Corollary</u>. The categories $\mathcal{R}^*(m, f)$ and $\mathcal{R}^*(r, f)$ are universal.

R e f e r e n c e s :

[1] A. PULTR and Z. HEDRLÍN, Relations (Graphs) with given infinite semigroups, to appear in Monatshefte für Mathematik.

[2] А. ПУЛЬТР, З. ГЕДРЛИН, О представлении малых категорий, submitted to Докл. АН СССР.