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Miroslav Katětov; Jiří Vaníček

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ON THE PROXIMITY GENERATED BY ENTIRE FUNCTIONS

M. KATĚTOV, J. VANÍČEK, Praha

We examine the proximity structure of the complex plane generated by the set of all entire functions. It is shown that this structure coincides with the finest proximity compatible with the usual topology of the plane.

In § 1, some fundamental concepts concerning proximity spaces are recalled, and the problems under consideration are formulated in terms of projectively generating mappings (however, this formulation is not used in what follows). In § 2, the problems in question are stated by means of current elementary topological concepts, In § 3, main theorems are stated, as well as some auxiliary propositions. Finally, § 4 contains the proofs.

§ 1 .

The simplest concepts of the theory of topological and uniform spaces are assumed to be known; however, for convenience, we recall certain concepts concerning proximity spaces (the theory of these spaces is due mainly to Yu.M. Smirnov [see, e.g. 5] ; a short survey of main concepts and results, as well as a list of references, is contained, e.g., in [1].

If M is a set, then a binary relation ϑ on the collection of all subsets of M is called a proximity structure (or simply a proximity) on M if, for any subsets X, Y, Z of M,

(1) $X \vartheta Y \Longleftrightarrow Y \vartheta X$,

(2) $(X \cup Y) \vartheta Z \Longleftrightarrow (X \vartheta Z$ or $Y \vartheta Z)$,

(3) $X \cap Y \neq \emptyset \Rightarrow X \vartheta Y$,

(4) $X \vartheta Y \Rightarrow X \neq \emptyset$,

(5) if X non ϑ Y , then there exist $X_1 \subset M$, $Y_1 \subset M$ such that $X_1 \cup Y_1 = M$ and neither $X \vartheta Y_1$ nor $Y \vartheta X_1$.

The pair (M, ϑ) is called a proximity space. If $X \vartheta Y$, we shall say that X and Y are near (under ϑ); if not, they are said to be distant (under ϑ). A mapping f of a proximity space (M, ϑ) into another one, (M_1, ϑ_1) , is called proximally continuous if $X \vartheta Y$ implies $f(X) \vartheta_1 f(Y)$. If ϑ , ϑ_1 are proximities on the same set M and $X \vartheta Y \Rightarrow$ $\Rightarrow X \vartheta_1 Y$, then we shall say that ϑ_1 is coarser than ϑ or that ϑ is finer than ϑ_1 . It is well known that if ϑ is a proximity on M , then the formula $x \in \overline{X} \Leftrightarrow (x) \vartheta X$ defines a completely regular topology on M ; we shall say that this topology is induced by the proximity ϑ .

We recall two simple instances of proximity spaces. If (M, ρ) is a metric space, then let ϑ be determined as follows: $X \vartheta Y$ if and only if for every $\varepsilon > 0$ there are points $x \in X$, $y \in Y$ with $\rho(x, y) < \varepsilon$; we shall say that ϑ is induced by the metric ρ . If M is a normal topological space, put $X \vartheta Y \Leftrightarrow \overline{X} \cap \overline{Y} \neq \emptyset$; then ϑ is the finest proximity inducing the prescribed topology of M .

We shall now introduce the concept of projective generation (for topological, uniform, and proximity spaces). It will not be used in what follows; however, its introduction may help to show the connection of problems considered here with certain notions of a quite general character.

Let us say, for convenience, t-space, u-space, p-space instead of topological (uniform, proximity) space. The terms t-, u-, p-continuous mapping, t-, u-, p-structure will be used in an analoguous way. Finally, the letter c will be used as a "variable" to be replaced by t or u or p .

Now let X be a set; for any $a \in A$, let f_a be a mapping of X into a c-space X_a. It is easy to show that there exists a coarsest c-structure on X under which every f_a is c-continuous; we shall say that this structure is projectively generated by the mappings f_a. An important special case is obtained if f_a are mappings into the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$ endowed with the usual structure (recall that the proximity and uniform structure of $\mathbb{R}$ or $\mathbb{C}$ are defined as follows: X and Y are near if $\inf_{x \in X, y \in Y} |x - y| = 0$; $\mathcal{G}$ is a uniform covering if there is a number $\varepsilon > 0$ such that for every point x there exists a $G \in \mathcal{G}$ with $|x - y| < \varepsilon \Rightarrow y \in G$).

It appears that the characterization of projectively generated proximity and uniform structures is not quite trivial even for some rather simple and natural sets of generators f_a. We shall consider the following two problems here.

Let H denote the set of all entire functions on $\mathbb{C}$ (i. e. of those functions $\varphi : \mathbb{C} \to \mathbb{C}$ which are holomorphic at every point $x \in \mathbb{C}$).

(I) To characterize the proximity on $\mathbb{C}$ projectively generated by H ; in particular, to decide whether it coincides with the finest proximity compatible with the usual topology of $\mathbb{C}$ (i.e. with the proximity under which X and Y are distant if and only if $\overline{X} \cap \overline{Y} = \emptyset$).

(II) To characterize the uniformity generated by H ; to decide whether it coincides with the finest uniformity compatible with the usual topology of $\mathbb{C}$.

The problem (I) is answered in the present note whereas problem (II) remains unsolved.

Clearly, there arise similar questions if we consider, instead of H , the class of all holomorphic mappings $f : \mathbb{C} \to E$ where E is locally convex topological complex linear space.

Finally, the above-mentioned problems are closely connected with the theory of Λ -structures introduced in [1] by one of the present authors. However, we shall not go into these questions here.

§ 2 .

Let δ and ν denote, respectively, the proximity and the uniformity generated by the set H of all entire functions.

Clearly, the structures δ and ν may be described as follows:

Two sets $X \subset \mathbb{C}$ and $Y \subset \mathbb{C}$ are near (under δ) if and only if, for any entire functions $f_1, \dots, f_n$ and any $\varepsilon > 0$, there exist points $x \in X$, $y \in Y$ with $|f_k(x) - f_k(y)| < \varepsilon$ for $k = 1, \dots, n$.

A collection $\mathcal{G}$ of subsets of $\mathbb{C}$ is a uniform covering of the space $(\mathbb{C}, \nu)$ if and only if there exist entire functions $f_1, \dots, f_n$ and a number $\varepsilon > 0$ with the following property: for every $x \in \mathbb{C}$ there is a set $G \in \mathcal{G}$ such that $y \in G$ whenever $|f_k(x) - f_n(y)| < \varepsilon$ for $k = 1, \dots, n$.

Problems (I) and (II) may now be reformulated as follows.

<u>Problem 1</u>. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed sets. To decide whether there exists a natural n (which may depend on X and Y) such that, for appropriate entire functions $f_1, \dots, f_n$, $\max_{1 \le k \le n} |f_k(x) - f_k(y)| \ge 1$ for any

$x \in X$, $y \in Y$.

<u>Problems 2 and 2'</u>. Let $\mathcal{G}$ be an open covering (for Problem 2': a finite open covering) of $\mathbb{C}$. To decide whether there exists a natural n (which may depend on $\mathcal{G}$) such that, for appropriate entire functions $f_1, \ldots, f_n$, the following holds: if $x \in \mathbb{C}$, $y \in \mathbb{C}$ and $\max_{1 \leq k \leq n} |f_k(x) - f_k(y)| < 1$, then there is a set $G \in \mathcal{G}$ with $x \in G$, $y \in G$.

It is well known that, for any two distant sets X and Y in a proximity space M , there exists a proximally continuous function h which separates X and Y in the sense that $h(z) = 0$ for $z \in X$, $h(z) = 1$ for $z \in Y$. It is also clear that, in the case of the space $\mathbb{C}$, h cannot be an entire function, in general (not even for a far weaker condition requiring that $x \in X \Longrightarrow |h(x)| < \varepsilon$, $y \in Y \Longrightarrow |h(y) - 1| < \varepsilon$).

Therefore, the following question seems to be natural:

<u>Problem 3</u>. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed sets, and let $G \subset \mathbb{C}$, $H \subset \mathbb{C}$ be disjoint open non-void. To decide whether there exists a natural number n (which may depend on X, Y, G, H) such that there exist entire functions $f_1, \ldots, f_n$ with the folllowing property: for any $x \in X$, $y \in Y$, there is a number $k = 1, \ldots, n$ with $f_k(x) \in G$, $f_k(y) \in H$.

To illustrate this problem, we are going to show that, for certain sets X, Y, G, H , $n = 2$ is not sufficient.

<u>Example</u>. Let $\{\alpha_k\}$ be an increasing sequence of positive numbers, $\alpha_k \to \infty$. Denote by T_k the set of all $x \in \mathbb{C}$ such that $|x| = \alpha_k$; let X and Y denote, respectively, the union of all T_k with k odd, and with k even. Let

$G \subset \mathbb{C}$, $H \subset \mathbb{C}$ be disjoint bounded open non-void.

Suppose that there are entire functions f_1, f_2 such that, for any $x \in X$, $y \in Y$, either $f_1(x) \in G$, $f_1(y) \in H$ or $f_2(x) \in G$, $f_2(y) \in H$. We may suppose that f_1, f_2 are not constant. For $k = 1,3,5,\ldots$, let A_k and B_k denote the set of those $x \in T_k$ for which $f_1(x) \in G$, respectively, $f_2(x) \in G$. Clearly, $T_k \subset A_k \cup B_k$, $k = 1,3,5,\ldots$. There exists an odd k_0 such that $B_k - A_k \neq \emptyset$; for otherwise $f_1(T_k) \subset G$ for $k = 1,3,5,\ldots$ which is a contradiction since G is bounded. Choose $x_0 \in B_{k_0} - A_{k_0}$. For any $y \in Y$, we obtain $f_1(x_0)$ non $\in G$, hence $f_2(x_0) \in G$, $f_2(y) \in H$; thus $f_2(y) \subset H$ which is a contradiction since H is bounded.

§ 3.

We may now state the main propositions. Observe that Theorem 1 solves Problem 3. As an immediate consequence, we obtain Theorem 2, which solves Problem 1; Theorem 3 (which solves Problem 2') also follows from Theorem 1. However, the solutions are not definitive; we do not know whether a smaller number of functions is sufficient.

<u>Theorem 1</u>. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed and let $G \subset \mathbb{C}$, $H \subset \mathbb{C}$ be disjoint open non-void. Then there exist entire functions $f_1,\ldots, f_9$ such that, for any $x \in X$, $y \in Y$, $f_k(x) \in G$, $f_k(y) \in H$ for some $k = 1,\ldots, 9$.

<u>Remark</u>. The example above shows that we cannot replace 9 with 2 in this assertion. On the other hand, we do not know whether 9 can be replaced by some $k = 3,\ldots, 8$.

<u>Theorem 2</u>. Let $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ be disjoint closed. Then there exist entire functions $f_1,\ldots, f_9$ such that, for any $x \in X$, $y \in Y$, $\max_{k=1,\ldots,9} |f_k(x) - f_k(y)| \geq 1$.

<u>Remark</u>. We do not know whether 9 can be replaced by

some $k = 1, \ldots, 8$. - If we denote by μ the metric proximity on $\mathbb{C}$ (i.e. the proximity under which X and Y are near if and only if $\inf_{x \in X, y \in Y} |x - y| = 0$), then Theorem 2 asserts that, for any $X \subset \mathbb{C}$, $Y \subset \mathbb{C}$ with $\overline{X} \cap \overline{Y} = \emptyset$, there exists a holomorphic mapping $f : \mathbb{C} \to (\mathbb{C}, \mu)^9$ such that $f(X)$ and $f(Y)$ are distant.

<u>Theorem 3</u>. Let $\mathcal{G}$ be an open cover of $\mathbb{C}$; let $\mathcal{G}$ consist of p sets. Then there exist entire functions $f_1, \ldots$ $\ldots, f_{9p}$ such that the following holds: if $x \in \mathbb{C}$, $y \in \mathbb{C}$ and $\max_{k=1,\ldots,9p} |f_k(x) - f_k(y)| < 1$ then there is a set $G \in \mathcal{G}$ with $x \in G$, $y \in G$.

<u>Remark</u>. In contradistinction to the preceding theorems, the number of functions given in this theorem depends on $\mathcal{G}$. We do not know whether this dependence is substantial or whether there is a number q with the following property: for any finite open cover $\mathcal{G}$ of $\mathbb{C}$, there exists a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}^q$ such that, with an appropriate $\varepsilon > 0$, $\|f(x) - f(y)\| < \varepsilon$ implies the existence of a set $G \in \mathcal{G}$ such that $x \in G$, $y \in G$.

The proof of Theorem 1 leans on two propositions from the theory of functions of a complex variable. The first of them is well known theorem of M.V. Keldysh [2] ; its proof is omitted. The second proposition is an easy consequence of the first (and a special case of some general theorems due to M.V. Keldysh and M.A. Lavrentiev [3] ; see also [4]).

<u>Proposition A</u>. If $E \subset \mathbb{C}$ is compact and $\mathbb{C} - E$ is connected, then for any complex-valued function f continuous on E and holomorphic on Int E , and any $\varepsilon > 0$, there exists a polynomial g such that $|f(z) - g(z)| < \varepsilon$ for $z \in E$.

Proposition B. Let $E \subset \mathbb{C}$ be closed an suppose that there exist compact sets $B_k \subset \mathbb{C}$, $k = 1,2,\ldots,$ such that

(1) $\cup B_k = \mathbb{C}$,

(2) for any $k = 1,2,\ldots,$ B_k is contained in $\operatorname{Int} B_{k+1}$ and does not intersect $\overline{E - B_k}$,

(3) the complement of $E \cap B_1$ as well as of every $B_k \cup \cup (E \cap B_{k+1})$, $k = 1,2,\ldots,$ is connected.

Then, for any complex-valued f continuous on E and holomorphic on $\operatorname{Int} E$ and any monotone positive function ϑ on reals $t \geq 0$, there exists an entire function g such that $|g(z) - f(z)| < \vartheta(|z|)$ for every $z \in E$.

In the proof of Theorem 1, the following assertion will be used.

Proposition C. If $D \subset \mathbb{C}$ and $S_i \subset \mathbb{C}$, $i = 1,\ldots, n$, are convex compact sets, and every two S_i, S_j, $i \neq j$, are disjoint, then $\mathbb{C} - D - \bigcup_{i=1}^{n} S_i$ is connected.

This proposition (in an essentially more general form) is well known. Its proof is omitted.

§ 4.

Proof of Proposition B. Let ε_k denote the greatest lower bound of $\vartheta(|z|)$ for $z \in B_k$. Then $\varepsilon_k \geq \varepsilon_{k+1} > 0$ for $k = 1,2,\ldots$. Let $\delta_k > 0$ be such that $\sum_{i=k}^{\infty} \delta_i < \varepsilon_k$, $k = 1,2,\ldots$. Put $B_0 = \emptyset$; let $g_0(z) = 0$ for every $z \in \mathbb{C}$.

By Proposition A, there exists a polynomial g_1 such that

$$|f(z) - g_1(z)| < \delta_1 \quad \text{for } z \in E \cap B_1 .$$

Now suppose that for a certain $m = 1,2,\ldots$ there are already chosen certain polynomials $g_1, g_2, \ldots, g_m$ such that

$$(*)\begin{cases} |f(z) - g_n(z)| < \delta_n & \text{for } z \in (E \cap B_n) - B_{n-1} \text{ and } 1 \leq n \leq m, \\ |g_n(z) - g_{n-1}(z)| < \delta_n & \text{for } z \in B_{n-1} \text{ and } 1 \leq n \leq m. \end{cases}$$

As a matter of fact, this has been done for $m = 1$. We shall now construct a polynomial g_{m+1} for which $(*)$ holds with $m + 1$ instead of m.

Put $\varphi(z) = 0$ for $z \in B_m$, $\varphi(z) = f(z) - g_m(z)$ for $z \in (E \cap B_{m+1}) - B_m$. Then φ is continuous on $B_m \cup (E \cap B_{m+1})$ and holomorphic in its interior. Since the complement of $B_m \cup (E \cap B_{m+1})$ is connected, there exists, by Proposition A, a polynomial h such that $|h(z)| < \delta_{m+1}$ for $z \in B_m$, $|f(z) - g_m(z) - h(z)| < \delta_{m+1}$ for $z \in (E \cap B_{m+1}) - B_m$.

Now put $g_{m+1} = g_m + h$. Then, clearly, $(*)$ holds with $m + 1$ instead of m. By induction, we obtain a sequence of polynomials $g_1, g_2 \ldots$ satisfying the inequalities $(*)$. Put $g(z) = \lim_{k \to \infty} g_k(z)$. It is easy to see that this sequence converges locally uniformly in $\mathbb{C}$; hence g is an entire function.

Clearly, if $z \in (E \cap B_k) - B_{k-1}$, $k = 1,2,\ldots,$ then $|f(z) - g(z)| < \sum_{i=k}^{\infty} \delta_i < \varepsilon_k$, and therefore $|f(z) - g(z)| < \vartheta(|z|)$.

Proof of Theorem 1. For any $\delta > 0$ let $\mathcal{S}(\delta)$ denote the collection of all squares with sides of length δ and vertices of the form $p\delta + iq\delta$ where p, q are integers. For $n = 1,2,\ldots$ let D_n denote the set of those $z \in \mathbb{C}$ for which $-n \leq \mathcal{R}(z) \leq n$, $-n \leq \mathcal{J}(z) \leq n$. Choose positive numbers δ_n, $n = 0,1,2,\ldots$ in such a way that δ_0^{-1} is an integer greater than 2, and

(1) for each $n = 0,1,2,\ldots$, $\delta_n = q_n \delta_{n+1}$ where q_n is an integer greater than 1, and

(2) if $n = 1,2,3,\ldots$, $x \in X \cap D_n$, $y \in Y \cap D_n$, then $\max(|\mathcal{R}(x - y)|, |\mathcal{J}(x - y)|) > 4\delta_{n-1}$.

Put $D_0 = \emptyset$ and, for $n = 1,2,\ldots,$ denote by $\mathcal{K}_n$ the collection of those squares $S \in \mathcal{S}(\delta_n)$ which are contained in $D_n - \operatorname{Int} D_{n-1}$. Put $\mathcal{K} = \bigcup \mathcal{K}_n$. Then $\mathcal{K}$ is a locally finite collection of compact sets and the following condition is fulfilled: if $S_i \in \mathcal{K}$, $i = 1, \dots, 4$, and $\bigcup_{i=1}^{4} S_i$ is connected, then either $X \cap \bigcup_{i=1}^{4} S_i = \emptyset$ or $Y \cap \bigcup_{i=1}^{4} S_i = \emptyset$.

We shall now construct three collections $\mathcal{K}^{(0)}, \mathcal{K}^{(1)}, \mathcal{K}^{(2)}$ of rectangles in the following way: $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(2)}$ consist of squares with sides parallel to the axes; a square belongs to $\mathcal{K}^{(0)}$ if and only if, for some n, the length of its side is equal to $\frac{1}{4}\sigma_n$, and its center x is a vertex of some $S_1 \in \mathcal{K}_n$, but of no square $S_2 \in \mathcal{K}_{n+1}$; a square belongs to $\mathcal{K}^{(2)}$ if and only if, for some n, the length of its side is equal to $\sigma_n - \frac{1}{8}\sigma_{n+1}$ and its centre coincides with the centre of some $S \in \mathcal{K}_n$; finally, it may be shown that the closure of $\mathbb{C} - \bigcup \mathcal{K}^{(0)} - \bigcup \mathcal{K}^{(2)}$ may be expressed as the union of a disjoint collection of rectangles, and this collection is taken as $\mathcal{K}^{(1)}$.

Obviously, the collection $\mathcal{K}^* = \mathcal{K}^{(0)} \cup \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)}$ has the following properties: (1) $\bigcup \mathcal{K}^* = \mathbb{C}$, (2) $\mathcal{K}^*$ is locally finite, (3) each $\mathcal{K}^{(j)}$ is a disjoint collection, (4) every $T \in \mathcal{K}^*$ is a compact convex set, (5) every $T \in \mathcal{K}^*$ is contained in the star (with respect to $\mathcal{K}$) of some $x \in \mathbb{C}$.

For $j = 0,1,2$, denote by $\mathcal{X}^{(j)}$ and $\mathcal{Y}^{(j)}$ the collection of those $T \in \mathcal{K}^{(j)}$ which intersect the set X (respectively, Y); let $X^{(j)}$ denote the union of all $T \in \mathcal{X}^{(j)}$, and similarly for $Y^{(j)}$; put $X^* = X^{(0)} \cup X^{(1)} \cup X^{(2)}$, $Y^* = Y^{(0)} \cup Y^{(1)} \cup Y^{(2)}$. Then $X \subset X^*$, $Y \subset Y^*$, $X^* \cap Y^* = \emptyset$, and $X^{(j)}$, $Y^{(j)}$ are closed. Choose points $a \in G$, $b \in H$; let $\varepsilon > 0$ be such that $|x - a| < \varepsilon$ implies $x \in G$, $|y - b| < \varepsilon$ implies $y \in H$; put $\varepsilon' = |b - a|^{-1} \varepsilon$.

To conclude the proof, it is now sufficient to find, for any given i, $j = 0,1,2$, and entire function $g = g_{ij}$ such that $|g(z)| < \varepsilon'$ for $z \in X^{(i)}$, $|g(z) - 1| < \varepsilon'$ for $z \in Y^{(j)}$. If such functions are constructed, then putting $f_{3i+j+1}(z) = a + (b - a)\, g_{ij}(z)$ we obtain functions $f_1, \dots, f_9$ with properties described in the theorem.

Now let i, j be given. Put $E = X^{(i)} \cup Y^{(j)}$ and denote by B_k the union of D_k and all those $T \in \mathcal{X}^{(i)} \cup \mathcal{Y}^{(j)}$ which intersect D_k. Then Proposition C implies that the assumptions from proposition B are fulfilled. Put $\vartheta(t) = \varepsilon'$ for $0 \leq t$, $f(z) = 0$ for $z \in X^{(i)}$, $f(z) = 1$ for $z \in Y^{(j)}$. By Proposition B, there exists an entire function g such that $|g(z) - f(z)| < \varepsilon'$ for every $z \in E$, hence $|g(z)| < \varepsilon'$ for $z \in X^{(i)}$, $|g(z) - 1| < \varepsilon'$ for $z \in Y^{(j)}$.

Proof of Theorem 3. Let $\mathcal{G}$ consist of sets $G_1, \dots, G_p$. Choose open sets V_i such that $\overline{V}_i \subset G_i$, $\bigcup_1^p V_i = \mathbb{C}$. By Theorem 2, there exist, for any $i = 1, \dots, p$, entire functions $f_{i,1}, \dots, f_{i,9}$ such that $\max_{k=1,\dots,9} |f_{i,k}(x) - f_{i,k}(y)| \geq 1$ whenever $x \in \overline{V}_i$, $y \in \mathbb{C} - G_i$. Consider the functions $f_{1,1}, \dots, f_{1,9}, \dots, f_{p,9}$. If $x \in \mathbb{C}$, $y \in \mathbb{C}$ and $|f_{i,j}(x) - f_{i,j}(y)| < 1$ for all $i = 1, \dots, p$, $j = 1, \dots, 9$, then, for some i, $x \in \overline{V}_i$ and therefore y does not belong to $\mathbb{C} - G_i$, hence $y \in G_i$. This concludes the proof.

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