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Some remarks on tensor products

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§ 1. A locally convex topology in $E \otimes F$.

Let $E$ and $F$ be two topological vector spaces over the field of real numbers. We shall define in the tensor product $E \otimes F$ a topology, which may be identified with the projective tensor topology (see [2]) in case, when $E$ and $F$ are locally convex spaces. We denote for a subset $A$ of $E$, $B$ subset of $F$, by $A \otimes B$ the set of all $x \otimes y \in E \otimes F$, where $x$ is in $A$, $y$ in $B$.

For any neighborhood $U$ of zero element in $E$, $V$ neighborhood of zero element in $F$ and for any positive integer $n$ we set

\begin{align*}
(1) & \quad K^n(U, V) = \mathbb{Z}^{\sum_{n=1}^{\infty} (U \otimes V + \ldots + U \otimes V)} (2^n \text{ summands on the right side}) \\
(2) & \quad \Omega_{U, V} = \bigcup_{n=1}^{\infty} K^n(U, V).
\end{align*}

The system of all $\Omega_{U, V}$, where $U$ varies in the neighborhood system $U$ of zero element in $E$, $V$ in neighborhood system $V$ in $F$, defines a topology on the tensor product $E \otimes F$. This topology is called in following discussion $\mathcal{Y}$-topology. It suffices to prove the relation $\Omega_{U, V} + \Omega_{U, V} \subseteq \Omega_{U, V}$, where $W \in U$, $V \in V$, $U + U \subseteq W$.

The proof of the last statement is obvious. $\mathcal{Y}$-topology in $E \otimes F$ is locally convex. In order to prove this fact it suffices to show the equality (see [3]) $\frac{1}{2} (\Omega_{U, V} + \Omega_{U, V}) = -\Omega_{U, V}$. This follows immediately from the definition (2).
The fundamental system of locally convex neighborhoods in $E \otimes F$ is formed by the collection of all interiors $\Omega^*, \nu$ of $\Omega, \nu$. The geometric significance of the neighborhoods $\Omega^*, \nu$ is clear: if we denote by $\omega(U \otimes V)$ the convex hull of $U \otimes V$ in $E \otimes F$, then $\Omega, \nu$ containing the interior of $\omega(U \otimes V)$ is contained in $\omega(U \otimes V)$. It follows at once that the closure of $\Omega^*, \nu$ is equal to the closed convex hull of $U \otimes V$ in $E \otimes F$. If $E$ and $F$ are locally convex spaces, then the equivalence of the $\mathcal{G}$-topology with the projective tensor topology (see [2]) follows from the inclusions:

$$\omega(U \otimes V) \subseteq \Omega^*, \nu \subseteq \Omega^*, \nu \subseteq \omega(U \otimes V),$$

where $U + U \subseteq U \otimes V$. We may define the $\mathcal{G}$-topology in $E \otimes F$ in the following manner, too; we set for any neighborhood $U$ of $0$ in $E$, $V$ in $F$ and any positive integer $n$:

1. $K^n(U, V) = \frac{U \otimes V + \ldots + U \otimes V}{n}$ (n summants on the right side).

2. $\Omega^*, \nu = \bigcup_{n=1}^{\infty} K^n(U, V)$.

It is clear that for any neighborhood $U$ in $E$, $V$ in $F$ holds

$$\Omega, \nu \subseteq \Omega^*, \nu \subseteq \omega(U \otimes V).$$

The last definition of the $\mathcal{G}$-topology can be acceptably generalized for tensor products of Abelian groups (see [6]). From (2) it follows at once that the natural bilinear mapping $(x, y) \rightarrow x \otimes y$ of $E \times F$ in $E \otimes F$ is continuous.

**Theorem 1.** Let $E$ and $F$ be two topological vector spaces. There exists a unique locally convex topology on the tensor product $E \otimes F$ having the following properties:

(a) the natural mapping $(x, y) \rightarrow x \otimes y$ of $E \times F$ in $E \otimes F$
is continuous on $E \times F$.

(b) If $G$ is a locally convex vector space, $f(x, y)$ a bilinear continuous mapping of $E \times F$ in $G$, the associated linear mapping $f^*$, defined by the algebraic isomorphism of the space $\mathcal{L}(E, F; G)$ of all bilinear mappings $E \times F \rightarrow G$ onto the space $\mathcal{L}(E \otimes F; G)$ of all linear mappings $E \otimes F \rightarrow G$, is continuous on $E \otimes F$.

Proof. The $G$-topology has properties (a) and (b). Indeed, (a) was established above, (b) follows from the fact that $f(U, V) \subseteq W$, where $U$, $V$, $W$ are neighborhoods of $0$ in $E$, $F$, $G$; implies $f^*(\Omega_U, V) \subseteq W$. The uniqueness of a topology having properties (a) and (b) is clear.

If $E$ is a topological vector space, $\tau$ a topology in $E$, then there exists a locally convex topology in $E$, coarser than $\tau$. For example the topology having as neighborhood of $0$ in $E$ the set $E$ only (the coarsest topology in $E$).

The supremum of all locally convex topologies in $E$, coarser than $\tau$, is a locally convex topology in $E$, coarser than $\tau$. We denote this topology by $\tau^*$. It is well-known that a linear function $f$ is continuous on $(E, \tau)$ if and only if $f$ is continuous on $(E, \tau^*)$. The topological dual of $E$ with the topology $\tau$ will be denoted by $E'$, $\sigma(E, E')$ means the weak topology in $E$ defined by $E'$, $\tau_0$ a topology in $E$ having as neighborhood the set $E$ only.

The following statements are equivalent:

(a) $E'$ contains an element different from zero-element,
(b) $\tau^*$ is larger than $\tau_0$,
(c) $\sigma(E, E')$ is larger than $\tau_0$.

Proof. If $0 \neq f \in E'$, then $\sigma(E, E')$ is larger than $\tau_0$, hence $\tau^*$ is larger than $\tau_0$. It suffices to prove (b)
implies (a). If (b) is satisfied, the factor space \( E/\mathcal{O} \), where \( \mathcal{O} \) is closure of zero-element in \((E, \tau^*)\) is separated locally convex space containing an element different from zero. Clearly \( E' \) has the property (a).

The topology \( \tau^* \) associated to \( \tau \) in \( E \) has as fundamental system of neighborhoods the collection of all \( \mathcal{O} \cup \mathcal{U} \), where \( \mathcal{O} \cup \mathcal{U} \) is the convex hull of \( \mathcal{U} \) in \( E, \mathcal{U} \) in \( E \).

**Proposition 1.** Let \( E \) and \( F \) be two topological vector spaces, \( \tau_1, \tau_2 \) topologies on \( E \) and \( F \). Then the \( \mathcal{O} \)-topology in \( E \otimes F \) defined by \( \tau_1 \) and \( \tau_2 \) is identical with the \( \mathcal{O} \)-topology in \( E \otimes F \) defined by \( \tau_1^* \) and \( \tau_2^* \).

Proof. In order to prove this proposition, it suffices to prove the inclusions \( \mathcal{O}_{\mathcal{U}},\mathcal{V} \subseteq \mathcal{O}_{\mathcal{U}},\mathcal{V} \subseteq \mathcal{O}(\mathcal{U} \oplus \mathcal{V}) \), where \( \mathcal{O} \mathcal{A} \) denotes the convex hull of \( \mathcal{A} \). It is evident that \( \mathcal{O}_{\mathcal{U}},\mathcal{V} \subseteq \mathcal{O}_{\mathcal{U}},\mathcal{V} \). Let \( x \in \mathcal{O}_{\mathcal{U}},\mathcal{V} \); there exist \( x_i \) in \( \mathcal{O}_{\mathcal{U}} \) (\( 1 \leq i \leq n \)) and \( y_j \) in \( \mathcal{O}_{\mathcal{V}} \) (\( 1 \leq i \leq n \)) such that

\[
x = \frac{1}{n} (x_1 \oplus y_1 + \ldots + x_n \oplus y_n).
\]

As \( x_i \) is in \( \mathcal{O}_{\mathcal{U}} \) (\( 1 \leq i \leq n \)), we may assume that \( x_i \) is of the form \( x_i = \lambda^i_1 x_1^i + \ldots + \lambda^i_{n_i} x_{n_i}^i \), where \( \sum_{k=1}^{n_i} \lambda^i_k = 1 \), \( \lambda^i_k \geq 0 \) (\( 1 \leq k \leq n_i \), \( 1 \leq i \leq n \)) and \( x_{n_i}^i \in \mathcal{U} \) (\( 1 \leq x \leq n_i \), \( 1 \leq i \leq n \)). Similarly \( y_j = \mu^j_1 y_1^j + \ldots + \mu^j_{m_i} y_{m_i}^j \), where \( \sum_{j=1}^{m_i} \mu^j_j = 1 \), \( \mu^j_j \geq 0 \) (\( 1 \leq j \leq m_i \), \( 1 \leq i \leq n \)) and \( y_{m_i}^j \in \mathcal{V} \) (\( 1 \leq j \leq m_i \), \( 1 \leq i \leq n \)).

The rest of the proof follows from the fact that

\[
x = \frac{1}{n} \left( \sum_{k=1}^{n} \lambda^k \sum_{j=1}^{m} \mu^j (x_k^j \oplus y_j^j) \right) \text{ is in } \mathcal{O}(\mathcal{U} \oplus \mathcal{V}).
\]

From the discussion it follows that \( \mathcal{O} \)-topology in tensor product \( E \otimes F \) of two topological vector spaces \((E, \tau_1), (F, \tau_2)\) is larger than the coarsest topology in \( E \otimes F \) if and only if \( \tau_1^* \) is larger than the coarsest topology in \( E \otimes F \).
$E$ and $\tau_2$ is larger than the coarsest topology in $F$.

Similarly $\tau_1$-topology in $E \otimes F$ is separated if and only if $\tau_1$ and $\tau_2$ are separated topologies in $E$ and $F$ respectively.

§ 2. $W$-topology in $E \otimes F$.

We assume as in § 1 $E$ and $F$ to be topological vector spaces, $U$ the system of all neighborhoods of $0$ in $E$, $V$ the system of all neighborhoods of $0$ in $F$. For any sequence $(U_i, i = 1, 2, \ldots)$, $U_i \in U$, and for any sequence $(V_i, i = 1, 2, \ldots)$, $V_i \in V$, we define

$$\Omega(U_i, V_i) = \{x \in E \otimes F; x \in \sum_i U_i \otimes V_i \} ,$$

where $\sum_i U_i \otimes V_i$ means the set of all $\sum_i x_i \otimes y_i + \ldots + x_n \otimes y_n$, $x_i \in U_i \ (1 \leq i \leq n)$, $y_i \in V_i \ (1 \leq i \leq n)$ an arbitrary integer.

If we choose $U'_i \in U$, $V'_i \in V \ (k = 1, 2, \ldots)$ satisfying

$$U'_i \subseteq U_{2k-1} \cap U_{2k}, \quad V'_i \subseteq V_{2k-1} \cap V_{2k} \ (k = 1, 2, \ldots)$$

then

$$\Omega(U'_i, V'_i) = \sum_i (U'_i \otimes V'_i) \subseteq \Omega(U_i, V_i).$$

The proof can be carried out easily by induction.

The natural bilinear mapping $(x, y) \rightarrow x \otimes y$ of $E \times F$ in $E \otimes F$ is continuous on $E \times F$.

Lemma. Let $G$ be a topological vector space, $W$ a neighborhood of zero element in $G$. We choose a neighborhood $W_i$ satisfying $W_i + W_i \subseteq W$. If $W_i (1 \leq i \leq k-1)$ are defined, we choose a neighborhood $W_k$ of $0$ such that

$$W_i + W_k \subseteq W_{i-1}$$

Then for any $k$ holds

$$W_i + W_k + \ldots + W_k + W_k \subseteq W \ ((k + 1) \text{ summands on the right side}) .$$

The proof can be carried out easily by induction.
Theorem 2. If \( f \) is a continuous bilinear mapping of \( E \times F \) in a topological vector space \( G \), then the associated linear mapping \( f^* \) of \( E \otimes F \) in \( G \) defined by
\[
(4) \quad f^*(x \otimes y) = f(x, y)
\]
is continuous on \( E \otimes F \). The correspondence \( f \leftrightarrow f^* \) defines an isomorphism of the space \( \mathcal{B}(E, F; G) \) of all bilinear continuous mappings \( E \times F \to G \) onto the space \( \mathcal{L}(E \otimes F; G) \) of all linear continuous mappings \( E \otimes F \to G \).

Proof. If \( f \in \mathcal{B}(E, F; G) \), \( W \) any neighborhood of zero element in \( G \), we choose \( U_i^* \) \( (i = 1, 2, \ldots) \) as in precedent lemma. For suitable neighborhoods \( U_i, V_i \) \( (i = 1, 2, \ldots) \) of zero element in \( E \) and \( F \) respectively we have \( f(U_i, V_i) \subseteq W_i \) \( (i = 1, 2, \ldots) \). The continuity of \( f^* \) follows from
\[
f^*(\bigcap U_i, \bigcap V_i) \subseteq W.
\]
For any \( x \in \bigcap U_i, \bigcap V_i \) there exist \( x_i \) in \( U_i \) \( (1 \leq i \leq n) \), \( y_i \) in \( V_i \) \( (1 \leq i \leq n) \) such that \( x = x_1 \otimes y_1 + \ldots + x_n \otimes y_n \). From
\[
f^*(x) = \sum_{i=1}^n f(x_i, y_i) \subseteq W_1^* + W_2^* + \ldots + W_n^* \subseteq W
\]
we derive \( f^*(\bigcap U_i, \bigcap V_i) \subseteq W \). This concludes the proof.

Consequence. On the tensor product \( E \otimes F \) \( W \)-topology is the unique topology compatible with the structure of a vector space satisfying following conditions:
(a) The natural bilinear mapping \( \langle x, y \rangle \to x \otimes y \) of \( E \times F \) in \( E \otimes F \) is continuous.
(b) If \( f(x, y) \) is a bilinear continuous mapping of \( E \times F \) in a topological vector space \( G \), then the linear mapping \( f^* \) associated to \( f \) is continuous on \( E \otimes F \).

The proof is evident.

If \( E \) and \( F \) are two finite dimensional separated topological spaces, then \( \mathcal{U}_G \)-topology on \( E \otimes F \) is identical
with the \( W \)-topology on \( E \otimes F \). This follows from ([7], theorem 26). Making use of theorem 2 we may conclude that \( W \)-topology on \( E \otimes F \) is larger or equal to \( \mathcal{Q} \)-topology on \( E \otimes F \). In general these topologies are different.

**Example.** Let \( (X, \mathcal{M}, \mu) \) be a measure space, \( \mathcal{M} \) a \( \sigma \)-algebra of subsets in \( X \), \( \mu \) a finite, atomic-free measure on \( \mathcal{M} \). By \( \mathcal{S} = \mathcal{S}(X, \mathcal{M}, \mu) \) we denote the space of all almost everywhere finite measurable functions on \( X \) with respect to \( \mathcal{M} \). We may define in \( \mathcal{S} \) a topology \( \tau \) by a metric \( \rho : \)

\[
\rho(f, g) = \int \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, d\mu.
\]

In \( \mathcal{S} \) there exists a unique open convex set, hence the associated locally convex topology \( \tau^* \) is the coarsest topology in \( \mathcal{S} \). For every \( \mathcal{A} \) there exists a decomposition \( (X_i)_{i \in J} \), \( J \) finite, \( X_i \cap X_j = \emptyset \) for \( i \neq j, j \in J, i \neq j, X_i \in \mathcal{M} (i \in J) \), \( X = \bigcup_{i \in J} X_i \) and \( \mu(X_i) = \frac{1}{|J|} \) (see [5]). For any \( f \in \mathcal{S} \) we set

\[
f_i(x) = \begin{cases} f(x), & x \in X_i, \\ 0, & x \notin X_i, \end{cases} \quad i \in J.
\]

Hence \( f_i \in \mathcal{S} (i \in J) \), \( f(x) = \sum_{i \in J} n_i f_i(x) \), where \( n \) is the cardinal of \( J \).

If we define a bilinear continuous mapping \( \mathcal{S} \times \mathcal{S} \to \mathcal{S} \) by \( (f, g) \to f \cdot g \), we may conclude that \( f \otimes g \to f \cdot g \) is a continuous mapping of \( \mathcal{S} \otimes \mathcal{S} \to \mathcal{S} \) with respect to the \( W \)-topology. From the proposition 1 of \& 1 it follows that linear mapping \( f \otimes g \to f \cdot g \) is not continuous in \( \mathcal{Q} \)-topology. This proves \( W \)-topology is larger than \( \mathcal{Q} \)-topology in \( \mathcal{S} \otimes \mathcal{S} \).
If $E$ and $F$ are $E'$- and $F'$-separated, $W$-topology in $E \otimes F$ is of course separated.

**Proposition 2.** Let $E_i \ (i = 1, 2)$, $F_i \ (i = 1, 2)$ be topological vector spaces, $u_i \ (i = 1, 2)$ a linear continuous mapping of $E_i$ in $F_i \ (i = 1, 2)$. Then the linear mapping $u_1 \otimes u_2$ of $E_1 \otimes F_1$ in $E_2 \otimes F_2$ defined by

$$\tag{5} (u_1 \otimes u_2)(x \otimes y) = u_1(x) \otimes u_2(y),$$

where $x \in E_1, y \in F_1$, is continuous on $E_1 \otimes F_1$.

The proof is evident.

**Proposition 3.** If we denote $E_i, F_i, u_i \ (i = 1, 2)$ as in proposition 2, then $u_1 \otimes u_2$ is an open mapping whenever $u_i \ (i = 1, 2)$ are open.

Proof. It suffices to prove

$$(6) \quad (u_1 \otimes u_2)\Omega(u_i, (u_i^c)), (v_i^c)) \subseteq \Omega(u_1(u_i), (u_2(v_i^c))),$$

where $U \subseteq U, V \subseteq V \ (i = 1, 2, \ldots)$.

For a given $x'$ in $E_2 \otimes F_2$ and $x'$ in $\Omega(u_i(u_i), (u_2(v_i^c)))$ we may choose suitable $x'_i \in u_i(u_i) (1 \leq i \leq n), y'_i \in u_2(v_i^c) (1 \leq i \leq n)$ satisfying $x' = x'_1 \otimes y'_1 + \ldots + x'_n \otimes y'_n$.

There exist $x_i \in U_i (1 \leq i \leq n), y_i \in V_i (1 \leq i \leq n), u_i(x_i) = x'_i (1 \leq i \leq n), u_2(y_i) = y'_i (1 \leq i \leq n)$. If we define

$$z = x_1 \otimes y_1 + \ldots + x_n \otimes y_n \quad \text{we have} \quad (u_1 \otimes u_2)(z) = x', \quad x \in \Omega(u_i, (v_i^c)),$$

**Proposition 4.** If $E_i \ (1 \leq i \leq n)$, $F_i$ are topological vector spaces, then the tensor product $\bigotimes_{i=1}^{n} E_i \otimes F$ is algebraic and topological isomorphic to $\bigotimes_{i=1}^{n} (E_i \otimes F)$.

Proof. The isomorphism $\varphi$ of $\bigotimes_{i=1}^{n} E_i \otimes F$ on $\bigotimes_{i=1}^{n} (E_i \otimes F)$ is given by (see [7])

$$\varphi((x_1 \otimes y), \ldots, (x_n \otimes y)) = (x_1, \ldots, x_n) \otimes y.$$

The continuity of $\varphi^{-1}$ follows from the continuity of
Proposition 5. Let $E$ and $F$ be two topological vector spaces, $M$ subspace in $E$, $N$ subspace in $F$. Then the factor space $E \otimes F / \Gamma(M,N)$ is algebraic and topological isomorphic to $(E/M) \otimes (F/N)$, where $\Gamma(M,N)$ is a subspace in $E \otimes F$ generated by the set of all $x \otimes y$, $x$ is in $M$ or $y$ is in $N$.

Proof. We denote by $\omega$, $\varphi$, $\psi$ the natural mapping of $E \otimes F$ into $E \otimes F / \Gamma(M,N)$. Then $E$ in $E/M$, $F$ in $F/N$.

The natural isomorphism $\Phi$ of $E \otimes F / \Gamma(M,N)$ on $(E/M) \otimes (F/N)$ is defined (see [1]) by

$$\Phi(\omega(x \otimes y)) = \varphi(x) \otimes \psi(y).$$

The continuity of $\Phi$ follows from the continuity of $x \otimes y \rightarrow \varphi(x) \otimes \psi(y)$ and $\Phi(\omega(\Gamma(M,N))) = 0$. From (6) we may conclude that $\Phi(\omega(\cup G_i), (\psi G_i)) \supseteq \cup \psi(\psi G_i))$.

This proves $\Phi$ is open.

§ 3. $W$-topology on the tensor product $G \otimes K$ of Abelian groups.

Let $G$ and $K$ be two Abelian topological groups written in the additive form, $G \otimes K$ their tensor product. Every Abelian group may be regarded as a module over the ring $\mathbb{Z}$ of all integers. By a $\mathbb{Z}$-linear ($\mathbb{Z}$-bilinear) mapping we mean a linear (bilinear) mapping of the module with respect to the ring $\mathbb{Z}$.

Every element $x$ of $G \otimes K$ is of the form

$$x = x_1 \otimes y_1 + \cdots + x_n \otimes y_n,$$

where $x_i \in G, (1 \leq i \leq n), y_i \in K, (1 \leq i \leq n)$ and $n$ is a positive integer.
For any sequence \((U^i, i = 1, 2, \ldots)\) of neighborhoods of zero in \(G\) and for any sequence \((V^j, j = 1, 2, \ldots)\) of neighborhoods of zero in \(K\) we define (as in § 2)\n
\[(3') \Omega_{(U^i), (V^j)} = \{ x \in G \otimes K, x \in \sum_{i \in I} U^i \otimes V^j \},\]

where \(\sum_{i \in I} U^i \otimes V^j\) means the set of all \(x_i \otimes y + \ldots + x_n \otimes y_n\), \(x_i \in U^j(1 \leq i \leq n), y_i \in V^j(1 \leq i \leq n)\) and \(n\) is any integer.

The collection of all \(\Omega_{(U^i), (V^j)}\) defines in \(G \otimes K\) a topology compatible with the structure of a group \(G \otimes K\).

This topology is called \(W\)-topology in \(G \otimes K\).

The proof of this statement is similar as in § 2. The natural \(Z\)-bilinear mapping \((x, y) \rightarrow x \otimes y\) of \(G \times K \rightarrow G \otimes K\) is continuous in \((0, 0)\). In general this mapping is not separated continuous on \(G \times K\). For example we may consider \(G\) additive group of the real numbers with the usual topology, \(K\) a discrete group with basis \((\xi_i)_{i \in J}\). The mappings \(x \rightarrow x \otimes \xi_i\) \((i \in J)\) are not continuous on \(G\).

**Theorem 3.** If \(f\) is a \(Z\)-bilinear mapping of \(G \times K\) in a topological group \(H\) continuous in \((0, 0)\), then the linear mapping \(f^*\) of \(G \otimes K\) in \(H\) associated to \(f\) and defined by

\[(4') f^*(x \otimes y) = f(x, y)\]

is continuous on \(G \otimes K\).

The proof is similar as in § 2.

**Consequence.** \(W\)-topology is the unique topology on \(G \otimes K\) compatible with the structure of a group and having the following properties:

(a) The natural \(Z\)-bilinear mapping \((x, y) \rightarrow x \otimes y\) is continuous in \((0, 0)\).

(b) If \(f\) is a \(Z\)-bilinear mapping of \(G \times K\) in a topological group \(H\) continuous in \((0, 0)\) then the associated
mapping $f^*$ defined by (4') is continuous on $G \otimes K$.

If $G$ or $K$ is a discrete group, then $G \otimes K$ is discrete. Hence, $W$-topology is equal to $\pi$-topology on $G \otimes K$ (see [6]), when $G \otimes K$ is torsion-free. From theorem 3 it follows that in general $W$-topology is larger or equal to $\pi$-topology.

Example. We denote by $D$ the ring of all $\mathfrak{p}$-adic numbers. Any element $x$ in $D$ is of the form $x = (k_1, k_2, \ldots, k_n, \ldots)$, where $k_n$ is a positive integer, $k_{n+1} = k_n \pmod{\mathfrak{p}^n}$, $0 \leq k_n < \mathfrak{p}^n$. A neighborhood $U_n(0)$ of zero element in $D$ is defined by (see [4])

$$U_n(0) = \{ x \in D; k_1 = \ldots = k_n = 0 \};$$

For any $x \in D$, $y \in D$ we set $f(x, y) = x \cdot y$. It is clear that $f$ is continuous in $\{0, 0\}$. The associated mapping $f^*$ is continuous on $D \otimes D$ with the $W$-topology. Making use of the fact that $\pi$-topology on $D \otimes D$ has as a neighborhood the set $D \otimes D$ only (see [6]), we may conclude that $f^*$ is not continuous on $D \otimes D$ with the $\pi$-topology. Hence, $W$-topology and $\pi$-topology are not identical.

If $G$ and $K$ are $(b)$-groups, (see [6]) (i.e. for any $x$ in $G$, $y$ in $K$ and any neighborhood $U$ of 0 in $G$, $V$ of 0 in $K$, there exist positive integers $m, n$ satisfying $x \in m U$, $y \in n V$), then $(x, y) \mapsto x \otimes y$ is continuous on $G \times K$ (see [6]).

Similarly as in § 2 for $W$-topology in the tensor product $G \otimes K$ of two Abelian groups hold propositions 2, 3, 4 and 5.

If $G$ and $K$ are two topological groups, then every character (see [4]) of $G \otimes K$ may be regarded as a $\mathbb{Z}$-bilinear mapping of $G \times K$ in $R/\mathbb{Z}$, where $R/\mathbb{Z}$ is the additive group of real numbers modulo 1 continuous in $(0, 0)$ and
conversely. Especially the natural $\mathbb{Z}$-bilinear mapping

$$(x, \chi) \rightarrow \langle x, \chi \rangle = \chi(x)$$

is a character of $G \otimes G^*$ where $G$ is a locally compact group, $G^*$ the group of all characters of $G$.

References:


