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Vlastimil Dlab

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THE RÔLE OF THE "FINITE CHARACTER PROPERTY" IN THE THEORY OF DEPENDENCE

V. DLAB, Praha

The purpose of this little note is to show some consequences of omitting the "finite character" axiom in an axiomatic dependence scheme. The note originated as a remark to one of Prof. R. Rado's problems mentioned in his lecture in the Conference on General Algebra in Warsaw, September 7-11,1964.

In order to avoid references to other papers we introduce, briefly, the basic concepts (in terms of the relation "an element depends on a set"). Let S be a set, $\mathfrak{P}S$ its power-set and $\rho \subseteq S \times \mathfrak{P}S$ a relation between elements and subsets of S. A subset $I \subseteq S$ is said to be ρ-independent if $[x, I \setminus (x)] \notin \rho$ for every $x \in I$; the family of all ρ-independent sets will be denoted by $\mathcal{I}_\rho$ ($\emptyset \in \mathcal{I}_\rho$ for any ρ). A relation ρ is called the dependence relation on S if it satisfies the following properties:

(I) $x \in X \rightarrow [x, X] \in \rho$ (incidence);

(E) $[x, X] \notin \rho \wedge [x, X \cup (y)] \in \rho \rightarrow [y, X \cup (x)] \in \rho$ (exchange);

(T) $[x, Y] \in \rho \wedge \forall y\, (y \in Y \rightarrow [y, X] \in \rho) \rightarrow [x, X] \in \rho$ (transitivity).

Let us remark that the property (T) together with (I) imply the following property (M) of a relation ρ

(M) $[x, Y] \in \rho \wedge Y \subseteq X \rightarrow [x, X] \in \rho$ (monotony).

Denote further by (E_k) and (T_k) the properties (E) and (T), respectively, restricted on $X \in \mathcal{I}_\rho$ and $Y \in \mathcal{I}_\rho$.

The following simple example of

$S_1 = (a, b, c)$ with $\rho_1 = (S_1 \times \mathcal{P} S_1) \setminus ([a,(b,c)], [b,(a,c)], [c,(a,b)])$

establishes the logical independence of (M) on (I), (E_k) and (T_k).

In paper [1], we have shown that all maximal ρ-independent sets (i.e. maximal elements of $\mathcal{I}_\rho$) have the same cardinality (the rank of S) if the relation ρ satisfies (I), (E_k), (T_k), (M) and

(F) $[x, X] \in \rho \rightarrow \exists F\, (F \subseteq X \wedge F$ finite $\wedge [x, F] \in \rho)$ (finite character) (i.e. ρ is a particular type of a GA-dependence relation introduced there). The main result of the present note reads that the same conclusion does not hold for a dependence relation ρ defined above. As a matter of fact, in this formulation the latter statement would be trivial; for, (I), (E) and (T) do not assure the existence of maximal elemets in $\mathcal{I}_\rho$ (this is a consequence of (F)), and the following example shows that no such elements may (in general) exist:

If S_2 is an infinite set and ρ_2 is defined by

$$[x, X] \in \rho_2 \leftrightarrow x \in X \quad \text{or} \quad X \quad \text{infinite},$$

then ρ_2 clearly satisfies (I), (E) and (T), and $\mathcal{I}_{\rho_2}$ being the family of all finite numbers of S_2 has no maximal elements.

To avoid this ambiguity in what follows we shall consider a dependence structure (S, ρ) as a pair of a set S and a dependence relation ρ with an additional property of

(B). $\mathcal{I}_\rho$ *has maximal elements* .

The main result reads then as follows.

<u>Theorem 1</u>. Let (S, ρ) be a dependence structure.

(i) If a maximal ρ-independent set is finite, then all are finite and have the same number of elements.

(ii) If a maximal ρ -independent set is infinite, then all are infinite.

It is evident that (ii) follows immediately from (i). The assertion (i) is then a consequence of the following two lemmas.

Lemma 1. Let ρ be a relation on S satisfying (I), (E_n), (T_n) and (M). Let M_1 and M_2 be two maximal ρ -independent sets and M_1 be finite. Then M_2 is finite, too.

Proof. Suppose, on the contrary, that M_2 is not finite. Let

$M_1 = (z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n)$, where $(z_1, z_2, \dots, z_m) = M_1 \cap M_2$;

evidently $n \geqslant 1$. Let us choose n elements of $M_2 \setminus M_1$ and denote by M_2' the (infinite) set of all remaining elements of $M_2 \setminus M_1$:

$$M_2 = (z_1, z_2, \dots, z_m, y_1, y_2, \dots, y_n) \cup M_2' .$$

Since $M_2 \setminus (y_1)$ is no longer maximal (however, in view of (M), it is ρ -independent), there is an element $x_{i_1} \in M_1$ such that

$$[x_{i_1}, M_2 \setminus (y_1)] \notin \rho ;$$

for, otherwise

$[y_1, M_1] \in \rho$ and $\forall x_i (x_i \in M_2 \to [x_i, M_2 \setminus (y_1)] \in \rho$ would, in view of (T_n), imply $[y_1, M_2 \setminus (y_1)] \in \rho$, a contradiction. Using (E_n) together with (M), we can easily verify that

$$M_{21} = (z_1, z_2, \dots, z_m, x_{i_1}, y_2, \dots, y_n) \cup M_2' \in \mathcal{J}_\rho .$$

Now, there is another element $x_{i_2} \in M_1$ such that

$$[x_{i_2}, M_{21} \setminus (y_2)] \notin \rho ;$$

this follows again from the fact that M_1 is maximal (and hence, $[y_2, M_1] \in \rho$). Thus

$$M_2 = (z_1, z_2, \dots, z_m, x_{i_1}, x_{i_2}, \dots, x_{i_n}, y_3, \dots, y_n) \cup M_2' \in \mathcal{J}_\rho .$$

Proceeding in this manner we reach in n steps the following ρ-independent set

$$M_{2n} = (z_1, z_2, \dots, z_m, x_{i_1}, x_{i_2}, \dots, x_{i_n}) \cup M_2' = M_1 \cup M_2' .$$

Hence, we get a contradiction of the maximality of M_1. The proof of Lemma 1 is completed.

The latter proof can be readily extended to finite sets M_1 and M_2 and we get thus

Lemma 2. Let ρ be a relation on S satisfying (I), (E_k), (T_k) and (M). If M_1 and M_2 are two finite maximal ρ-independent sets, then they have the same number of elements.

P r o o f . Since both

$$\mathrm{card}(M_1) \geq \mathrm{card}(M_2) \quad \text{and} \quad \mathrm{card}(M_1) \leq \mathrm{card}(M_2),$$

Lemma 2 immediately follows.

The following theorem shows that (ii) of Theorem 1 cannot be strengthened.

Theorem 2. Let $(\alpha_\gamma)_{\gamma \in \Gamma}$ be a family of infinite cardinal numbers. Then there exists a dependence structure with a family $(M_\gamma)_{\gamma \in \Gamma}$ of maximal independent sets such that

$$\mathrm{card}(M_\gamma) = \alpha_\gamma \quad \text{for each } \gamma \in \Gamma .$$

P r o o f . Consider a family $(S_\gamma)_{\gamma \in \Gamma}$ of mutually disjoint sets such that

$$\mathrm{card}(S_\gamma) = \alpha_\gamma \text{ for each } \gamma \in \Gamma ,$$

and denote by S_0 the union of these sets $S_0 = \bigcup_{\gamma \in \Gamma} S$. Define the relation $\rho_0 \subseteq S_0 \times \mathcal{P} S_0$ on S_0 in the following way: For $x \in S_0$ and $X \subseteq S_0$,

$$(*) \quad [x, X] \in \rho_0 \leftrightarrow x \in X \quad \text{or, for a certain } \gamma_0 \in \Gamma,$$
$$X = (S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}, \text{ where } F_{\gamma_0} \subseteq S_{\gamma_0} \text{ is finite,}$$
$$A_{\gamma_0} \subseteq \bigcup_{\substack{\gamma \in \Gamma \\ \gamma \neq \gamma_0}} S_\gamma \quad \text{and } \mathrm{card}(A_{\gamma_0}) \geq \mathrm{card}(F_{\gamma_0}) .$$

It can be easily seen that, besides (I), also (T) is satisfied by this relation ρ_0. Now, prove the validity of (E) for ρ_0. Thus, let $x \in S_0$, $y \in S_0$ and $X \subseteq S_0$ be such that

$(**)$ $\quad [x, X] \notin \rho_0$ and $[x, X \cup (y)] \in \rho_0$.

Then, $x \notin X$. The conclusion $[y, X \cup (x)] \in \rho_0$ is trivial for $x = y$; suppose, therefore, that $x \neq y$. The assumption $(**)$ implies that

$X \cup (y) = (S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}$ with $card\,(F_{\gamma_0}) = card\,(A_{\gamma_0}) < \aleph_0$.

for a suitable $\gamma_0 \in \Gamma$. We have to consider four (in fact, very similar) cases:

(i) $y \in S_{\gamma_0}$, $x \in S_{\gamma_0}$, i.e. $y \in S_{\gamma_0} \setminus F_{\gamma_0}$, $x \in F_{\gamma_0}$; then, evidently, $[y, (S_{\gamma_0} \setminus [(F_{\gamma_0} \cup (y)) \setminus (x)]) \cup A_{\gamma_0}] \in \rho_0$;

(ii) $y \in S_{\gamma_0}$, $x \in S_{\gamma_0}$, i.e. $y \in S_{\gamma_0} \setminus F_{\gamma_0}$, $x \notin S_{\gamma_0} \cup A_{\gamma_0}$; then,

$[y, (S_{\gamma_0} \setminus [F_{\gamma_0} \cup (y)]) \cup A_{\gamma_0} \cup (x)] \in \rho_0$;

(iii) $y \notin S_{\gamma_0}$, $x \notin S_{\gamma_0}$, i.e. $y \in A_{\gamma_0}$, $x \in F_{\gamma_0}$; then,

$[y, (S_{\gamma_0} \setminus [F_{\gamma_0} \setminus (x)]) \cup (A_{\gamma_0} \setminus (y))] \in \rho_0$;

(iv) $y \notin S_{\gamma_0}$, $x \notin S_{\gamma_0}$, i.e. $y \in A_{\gamma_0}$, $x \notin S_{\gamma_0} \cup A_{\gamma_0}$; then,

$[y, (S_{\gamma_0} \setminus F_{\gamma_0}) \cup ([A_{\gamma_0} \setminus (y)] \setminus (x))] \in \rho_0$.

Thus, (E) holds for ρ_0.

Moreover, since, for any element $x \in S_\gamma$, $S_\gamma \setminus (x)$ is not of the form described in $(*)$, S_γ is ρ_0-independent for each $\gamma \in \Gamma$. Also, $S_\gamma = (S_\gamma \setminus \emptyset) \cup \emptyset$ is maximal for each $\gamma \in \Gamma$, hence, the last condition (B) is satisfied for ρ_0 and, thus, (S_0, ρ_0) is a dependence structure (in the sense of this note).

This completes the proof, for the existence of maximal ρ_0-independent sets with prescribed cardinalities has also been established (take e.g. $M_\gamma = S_\gamma$).

R e m a r k . As a matter of fact, referring back to the dependence structure (S_0, ρ_0) constructed in the proof of

Theorem 2, all sets of the form

$(* * *)\quad (S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}$ with $A_{\gamma_0} \cap S_{\gamma_0} = \emptyset$ and $card(F_{\gamma_0}) = card(A_{\gamma_0}) < \aleph_0$ are maximal and ρ_0-independent. Evidently,

$$card((S_{\gamma_0} \setminus F_{\gamma_0}) \cup A_{\gamma_0}) = card(S_{\gamma_0}) = \alpha_{\gamma_0}$$

On the other hand, any maximal ρ_0-independent set of this structure is of the form $(* * *)$. For, any maximal set must necessarily be of the form $(*)$ and any maximal ρ_0-independent set must, moreover, satisfy the last condition on cardinalities in $(* * *)$. Thus, the cardinality of an arbitrary maximal ρ_0-independent set of (S_0, ρ_0) is equal to one of the numbers

Finally, let us remark that the maximal ρ_0-independent sets of (S_0, ρ_0) satisfy also the conditions denoted in [2] by $(\tilde{B}'_{2f})$ and $(\tilde{B}_{2f})$:

(B'_{2f}) For any two maximal independent sets M_1 and M_2 and any finite subset $M'_1 \subseteq M_1 \setminus M_2$ there exists a subset $M'_2 \subseteq M_2 \setminus M_1$ of the same number of elements such that $(M_1 \setminus M'_1) \cup M'_2$ is a maximal independent set.

$(\tilde{B}'_{2f})$ For any two maximal independent sets M_1 and M_2 and any finite subset $M'_1 \subseteq M_1 \setminus M_2$ there exists a subset $M'_2 \subseteq M_2 \setminus M_1$ of the same number of elements such that $M'_1 \cup (M_2 \setminus M'_2)$ is a maximal independent set.

Both properties suffice to be proved for single-point subsets M'_1 and M'_2 (the properties (B'_2) and $(\tilde{B}'_2)$ in [2]); the proof involves several simple cases to be considered and is left to the reader. Thus, the example of the dependence structure (S_0, ρ_0) in the proof of Theorem 2 shows that the assumption of the finite character property

(B_3) If every finite subset of a set X is a subset of a suitable maximal independent set, then X is a subset of a maximal independent set. was essential in § 5 of [2].

In order to show also the logical independence of (B_3) on the stronger properties (B'_{2q}) and $(\tilde{B}'_{2q})$ of [2], consider the following simple example (S_*, ρ_*) of a dependence structure:

$$S_* = S_1 \cup S_2 \quad \text{with } S_1 \cap S_2 = \emptyset, \; card(S_1) \geq \aleph_0, \; card(S_2) > \aleph_0 .$$

and

$$[x, X] \in \rho_* \longleftrightarrow x \in X \quad \text{or} \quad card(X \cap S_2) \geq \aleph_0 \quad \text{or}$$
$$X = (S_1 \setminus F_1) \cup A_2 \text{ with } F_1 \text{ finite and } card(F_1) \leq card(A_2) .$$

It is a matter of routine to check that ρ_* satisfies (I), (E) and (T), that all maximal ρ_* -independent sets are of the form

$$M = (S_1 \setminus F_1) \cup A_2 \quad \text{with } card(F_1) = card(A_2) < \aleph_0$$

and that they satisfy the properties (B'_{2q}) and $(\tilde{B}'_{2q})$ (which reduce to (B'_{2f}) and $(\tilde{B}'_{2f})$, respectively). All maximal ρ_* -independent sets have thus the same cardinality ($= card(S_1)$) – a fact which follows, in general, from the property (B'_{2q}). However, it turns out that (B_3) is not fulfilled:

Let T be a countable subset of S_2 and $(F_{1n})_{n \geq 1}$ a family of subsets of S_1 such that

$$card(F_{1n}) = n \quad \text{for every } n \geq 1 .$$

Then, for any finite subset F_2 of T there is a natural number n (the number of elements of F_2) such that

$$(S_1 \setminus F_{1n}) \cup F_2$$

is a maximal ρ_* -independent set. But, there is no maximal

ρ_* -independent set containing the set T.

R e f e r e n c e s

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[2] V. DLAB, Axiomatic treatment of bases in arbitrary in arbitrary sets, to appear in Czech.Math.J.