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Commentationes Mathematicae Universitatis Carolinae, Vol. 6 (1965), No. 2, 149--155

Persistent URL: http://dml.cz/dmlcz/105005

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6, 2 (1965)

A RIGID RELATION EXISTS ON ANY SET

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The aim of this note is to prove that, for any set X, there exists a binary relation $R \subset X \times X$ such that the identity transformation is the only mapping $f: X \rightarrow X$ for which the implication

$$x R y \Rightarrow f(x) R f(y)$$

holds. Moreover, we are going to show some consequences of this assertion.

First, the definitions and notation.

Saying "a relation on a set X " we mean always a binary relation, i.e. a subset of $X \times X$. Let R (S , resp.) be a relation on a set X (Y , resp.). If $f: X \rightarrow Y$ and

$$x R y \Rightarrow f(X) S f(y) \quad \text{for all} \quad x, y \in X ,$$

f is called an RS -compatible mapping and we write $f: (X, R) \rightarrow (Y, S)$.

$C(X, R)$ denotes the semigroup, under composition, of all compatible mappings from (X, R) into itself. (X, R) is said to be rigid, if $C(X, R)$ is trivial.

Using the above definitions our aim is to prove the headline.

The following assertion - denoted by $\mathcal{F}(\mathfrak{m})$, $\mathfrak{m}$ a cardinal-played an important role in a few theorems (see [1], [2], [3], [4], [5], [6]): There exists a rigid (X, R) such that $\operatorname{card} X > \mathfrak{m}$.

$\mathcal{F}(\mathfrak{m})$ was proved in [4] for $\mathfrak{m}$ less than the first inaccessible cardinal. It follows from the result of the present note that $\mathcal{F}(\mathfrak{m})$ holds for every cardinal $\mathfrak{m}$. It enables to omit unpleasant assumptions concerning accessibility of cardinals in the mentioned papers.

Construction.

If α, β are ordinals, we use the symbols $\alpha \prec \beta$, $\alpha \succ \beta$, $\alpha \leq \beta$, $\alpha \geq \beta$ in the ordinary sense. The ordinal 0 is considered as a limit ordinal. Let ω_ξ be the least ordinal with $card\ \omega_\xi = \aleph_\xi$.

Put $D = \{\alpha \mid \alpha \leq \omega_\xi + 1\}$. Let D_0 denote the set of all limit ordinals in D which are confinal with ω_0; by D_1 we denote the set of all limit ordinals in D which are not confinal with ω_0; finally, by D_2 we denote the set of all non-limit ordinals in D. Evidently, $D = D_0 \cup D_1 \cup D_2$ and D_0, D_1, D_2 are mutually disjoint.

If $\alpha \in D_0$, we choose an increasing sequence $\{\alpha_n \mid n \geq 2\}$ such that α is its supremum and $\alpha_n = \bar{\alpha}_n + n$, where $\bar{\alpha}_n$ is a limit ordinal (the $\bar{\alpha}_n$'s need not be different for different n). We emphasize that the symbols β_n and $\bar{\beta}_n$ will be used always in this sense.

We define a relation R on D as follows:

(1) $0 R 2$,

(2) $\alpha R (\alpha + 1)$ for all $\alpha \leq \omega_\xi$,

(3) if $\beta \in D_1$, $\alpha R \beta$ if and only if $\alpha \prec \beta$ and $\alpha \in D_0 \cup D_1$,

(4) if $\alpha \in D_0$, $\gamma R \alpha$ if and only if $\gamma = \alpha_n$ for some $n \geq 2$,

(5) $\alpha \, R \, (\omega_\xi + 1)$ if and only if either $\alpha = \omega_\xi$ or $\alpha \in D_2 \setminus \{\omega_\xi + 1\}$.

Remarks. 1) Evidently, $\alpha \, R \, \beta$ implies $\alpha < \beta$.

2) $\beta \, R \, 2$ if and only if $\beta = 0$ or $\beta = 1$. If $\alpha \in D_2 \setminus \{2, \omega_\xi + 1\}$, then $\beta \, R \, \alpha$ if and only if $\alpha = \beta + 1$.

3) If $\beta \, R \, (\omega_\xi + 1)$ and $\beta \in D_0 \cup D_1$, then $\beta = \omega_\xi$.

Further, we shall always assume that $f \in C(D, R)$.

Lemma 1. $\alpha < \beta$ implies $f(\alpha) < f(\beta)$. In particular, f is a one-to-one mapping.

Proof. Denote by β the least ordinal for which the assertion does not hold. Choose - once for all - an ordinal α such that $\alpha < \beta$ and $f(\alpha) \geq f(\beta)$.

a) Let $\beta \in D_1$. Then there exists an ordinal $\gamma \in D_0 \cup D_1$ such that $\alpha < \gamma < \beta$ (it suffices to choose $\gamma = \sup \{\alpha + n \mid n = 0, 1, \dots\}$). Hence, $f(\alpha) < f(\gamma)$ and, by $\gamma \, R \, \beta$, $f(\gamma) < f(\beta)$ - a contradiction.

b) Let $\beta \in D_0$. Then $\alpha < \beta_n < \beta$ for some natural n. Hence, $f(\alpha) < f(\beta_n)$. Since $\beta_n \, R \, \beta$, we have $f(\beta_n) \, R \, f(\beta)$ and $f(\beta_n) < f(\beta)$ - a contradiction.

c) Let $\beta \in D_2$. Then $\beta = \beta' + 1$ and $\alpha \leq \beta' < \beta$. Hence, $f(\alpha) \leq f(\beta')$ and, by $\beta' \, R \, \beta$, $f(\beta') < f(\beta)$ - a contradiction.

Lemma 2. $f(\alpha) \geq \alpha$ for every $\alpha \in D$. In particular, $f(\omega_\xi + 1) = \omega_\xi + 1$ and $f(\omega_\xi) = \omega_\xi$.

Proof. Let $f(\alpha) < \alpha$. By lemma 1, we get easily $f^{k+1}(\alpha) < f^k(\alpha)$. Hence, the sequence

$\{f^k(\alpha)\}$ is decreasing in a contradiction with the well ordering of D. $f(\omega_\xi) = \omega_\xi$ follows from the fact that f is a one-to-one mapping.

Lemma 3. If $\alpha \in D_2$ ($\alpha \in D_2$ and $\alpha \neq \omega_\xi + 1$, resp.), then $f(\alpha) \in D_2$ ($f(\alpha) \in D_2$ and $f(\alpha) \neq \omega_\xi + 1$).

Proof. The assertion is evident for $\alpha = \omega_\xi + 1$. If $\alpha \in D_2$, $\alpha \neq \omega_\xi + 1$, we have $\alpha R (\omega_\xi + 1)$. Hence, $f(\alpha) R (\omega_\xi + 1)$. If $f(\alpha) \notin D_2 \setminus \{\omega_\xi + 1\}$, then, by 3) in the remark, $f(\alpha) = \omega_\xi$. It is impossible, as f is one-to-one and $f(\omega_\xi) = \omega_\xi$.

Lemma 4. If $\alpha + n \in D$, n natural, then $f(\alpha + n) = f(\alpha) + n$.

Proof. Let $\alpha = 0$. Then $0R2$, $1R2$ and $f(0)Rf(2)$, $f(0)Rf(2)$. $f(2) \neq \omega_\xi + 1$, as f is one-to-one. If $f(2) \neq 2$, by lemma 3 and 2) in the remark, we get $f(0) = f(1)$. Hence, $f(2) = 2$. As $1R0$ does not hold, we get $f(0) = 0$, $f(1) = 1$.

Let $n > 2$. Then there is only one $\beta \in D_2$ such that $nR\beta$, namely, $\beta = n + 1$. By induction, we get easily $f(n) = n$. Thus, the assertion holds for $\alpha = 0$, and, moreover, we see that it holds for any finite α.

Evidently, the assertion holds for $\alpha = \omega_\xi$, $\omega_\xi + 1$.

Let α be an infinite ordinal, $\alpha \neq \omega_\xi$, $\omega_\xi + 1$. It suffices to prove that $f(\alpha + 1) = f(\alpha) + 1$. Obviously, $f(\alpha) R f(\alpha + 1)$ and $f(\alpha + 1) \in D_2$. As f is one-to-one $f(\alpha + 1) \neq 0, 2$. Hence, by 2) in the remark, $f(\alpha + 1) = f(\alpha) + 1$.

Lemma 5. $\alpha \in D_0 \cup D_1$ implies $f(\alpha) \in D_0 \cup D_1$.

Proof. Evident for $\alpha = 0$. If $\alpha > 0$, there are infinitely many ordinals γ such that $\gamma R \alpha$ and, since f is one-to-one, infinitely many δ such that $\delta R f(\alpha)$. As $f(\alpha) \neq \omega_\xi + 1$, $f(\alpha)$ must belong to $D_0 \cup D_1$.

Lemma 6. If $\alpha \in D_0$, then $\beta = f(\alpha) \in D_0$. Moreover, $f(\alpha_n) = \beta_n$.

Proof. Since $\alpha_n R \alpha$, $f(\alpha_n) = (f(\bar{\alpha}_n) + n) R \beta$. If $\beta \in D_1$, then $f(\alpha_n) \in D_0 \cup D_1$, which is a contradiction. $\beta \notin D_2$, by lemma 5. Hence, $\beta \in D_0$ and $f(\alpha_n) = \beta_k$ for some natural k, i.e. $f(\bar{\alpha}_n) + n = \bar{\beta}_k + k$. As $f(\bar{\alpha}_n), \bar{\beta}_k \in D_0 \cup D_1$, we get $k = n$.

Theorem 1. (D, R) is rigid.

Proof. Let $\alpha^1 \in D$, $f(\alpha^1) \neq \alpha^1$. By lemma 2, $f(\alpha^1) > \alpha^1$. By lemma 1, $f^n(\alpha^1) < f^{n+1}(\alpha^1)$ for all natural n. Put $\alpha^n = f^{n-1}(\alpha^1)$ for $n \geq 2$, $\alpha = sup\{\alpha^n \mid n = 0, 1, 2, ..\}$. Evidently, $\alpha \in D_0$. Let $f(\alpha) = \beta > \alpha$. Then $\alpha < \beta_n < \beta$ for some natural n. Moreover, there is a natural i such that $\alpha_n < \alpha^i < \alpha$. Hence, $\beta_n = f(\alpha_n) < f(\alpha^i) = \alpha^{i+1} < \alpha$, a contradiction. We get $f(\alpha) = \alpha$.

By lemma 6, $f(\alpha_n) = \alpha_n$. As $\alpha^1 < \alpha$, $\alpha^1 < \alpha_n < \alpha$ for some natural n. Hence, $\alpha^2 = f(\alpha^1) < \alpha_n < \alpha$, and, by induction, $\alpha^i < \alpha_n$ for all natural i. Finally, $\alpha = sup\{\alpha^i\} \leq \alpha_n < \alpha$ - a contradiction.

Theorem 2. For any set X, there exists a rigid relation R on X. Moreover, we may choose $R \subset R'$, where R' is a well ordering of X.

Proof. Every strict ordering of a finite set is rigid. Let X be infinite, $\operatorname{card} X = \aleph_\xi$. Construct (D, R) for ω_ξ. (D, R) is rigid and R is a subset of the relation of well ordering. As $\operatorname{card} D = \operatorname{card} X$, there is a one-to-one mapping of D onto X. This mapping enables us to define a rigid relation on X with the required property.

Consequences.

I. Every semigroup S^1 with a unity element is isomorphic with $C(X, R)$ for a set X and a relation $R \subset X \times X$. If S^1 is finite, X may be chosen finite or of any infinite cardinal. If S^1 is infinite, $\operatorname{card} X$ may be arbitrary cardinal greater or equal to $\operatorname{card} S^1$.

The proof follows from [4], as $\mathcal{F}(\mathfrak{m})$ holds for any cardinal. The assertions concerning cardinals can be obtained easily considering the constructions in [4] and theorem 2.

II. The last assertions hold, if we consider only symmetric relations.

If follows from I. and [3].

III. If X is an infinite set, then there exists a rigid symmetric relation on X.

The proof follows from theorem 2 and [3].

IV. Denote by $\mathfrak{R}$ the category of all couples (X, R), X is a set and $R \subset X \times X$ and their compatible mappings.

$\mathfrak{R}$ is universal (i.e. every small category is isomorphic with a full subcategory of $\mathfrak{R}$). Similarly, the categories defined in [1],[5],[6]. We had to assume in the quoted papers that we work in a set theory without inaccessible cardinals. Now, we may omit this assumption.

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