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HOMOLOGICAL FIXED POINT THEOREMS III.

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Theorem 2 of [4] asserts the presence of a fixed point for some one of the iterates $f, f^2, \ldots, f^m$ of any continuous map $f: X \to X$, under rather strict restrictions on X (non-oddness, l.c.; one may then take $m = \chi(X)$). In some cases it may be useful to weaken the conditions on X but restrict the maps f considered. Indeed, some theorems of this type are already known: assume X triangulable; if $\chi(X) \neq 0$ and $f: X \to X$ is homotopic to the identity map, then f has a fixed point (a corollary to the Hopf-Lefschetz fixed point theorem); or, more generally, if $f: X \to X$ is homotopic to a retraction $X \to Y$ with $\chi(Y) \neq 0$, then again f has a fixed point (theorem 6 in [3]). The main result of this paper, theorem 4, is another result of this type. In particular, it is shown that if $f: X \to X$ is homologuous to a homeomorphism and $\chi(X) \neq 0$, then some iterate f^s has a fixed point (and an upper bound to s is given: corollary 5).

The terminology and notation of [3] are preserved. In particular, "group" means an abelian group G with fixed integrity domain J as left operators, and with finite rank over J (this rank will now be denoted by $\pi(G)$). A "group sequence" is a sequence $\{G_q\}$ of such groups with $\prod G_q$ again of finite rank over J. The Euler characteristic of $G = \{G_q\}$ is, as in [3], defined as

$$\chi(G) = \Sigma(-1)^q \pi(G_q).$$

For p, j, gli we refer to definitions 1 to 3 in [3]. It seems useful to introduce the following notation

__Definition 1.__ For a group sequence $G = \{G_q\}$ set

$$\varkappa(G) = \Sigma \pi(G_q).$$

Obviously $\varkappa(G) = \pi(\Pi G_q), \varkappa(G) \geq 0$, and $\varkappa(G) = 0$ iff all G_q are periodic; $\chi(G) \leq \varkappa(G)$, with equality iff all odd-indexed G_{2q+1} are periodic; $\varkappa(G) \pm \chi(G)$ are both even integers.

For triangulable spaces X (i.e. topological spaces with a finite triangulation), the homology sequence is denoted by $H_*(X) = \{H_q(X)\}$, the q-th Betti number by $\pi_q(X) = \pi(H_q(X))$; $f_* : H_*(X_1) \to H_*(X_2)$ denotes the homomorphism induced by a continuous $f : X_1 \to X_2$; and we define

$$\chi(X) = \chi(H_*(X)) = \Sigma(-1)^q \pi_q(X), \quad \varkappa(X) = \varkappa(H_*(X)) = \Sigma \pi_q(X).$$

In particular, $\chi(X) = \varkappa(X)$ iff X is non-odd (cf. the definition in [4, p.87]). (As another example, for compact 2-manifolds X, $\varkappa(X) - \chi(X) = 4 \times$ (genus of X.)

__Lemma 2.__ If $f : G \to G$ is a homomorphism of a group G, then

$$j(f;\lambda) = \Sigma_{k=1}^{n} \frac{\lambda_k}{1-\lambda\lambda_k}$$

with $n = \mathit{rank}\ \mathit{im} f \leq \pi(G)$, and $0 \neq \lambda_k$ in a root field over J. If $f : G \to G$ is a homomorphism of a group sequence G, then

$$gli(f;\lambda) = \Sigma_{k=1}^{n} \frac{r_k \lambda_k}{1-\lambda\lambda_k} \tag{1}$$

with distinct $\lambda_k \neq 0$, integers r_k and

$$\Sigma_1^n r_k = -\chi(\mathit{im} f), \qquad n \leq \varkappa(G).$$

(Proof.) In the definition of $p(\cdot)$ as

$$p(f;\lambda) = det(I - \lambda D^{-1}A)$$

obviously

$$degree\ p = rank\ D^{-1}A = rank\ A = \pi(im\ f).$$

Hence the decomposition of $p(\cdot)$ in its root field J_p over J may be written as (cf.proof of theorem 2 in [3]; note that $p(f; 0) = det\ I = 1$)

$$p(f;\lambda) = \Pi_{k=1}^{n}(1-\lambda\lambda_k)$$

with $0 \neq \lambda_k \in J_p$, $n = \pi(im\ f)$. Then $j = -\frac{dp}{d\lambda}/p$ yields the first assertion.

The second then results on applying the first to

$$gli(f, \lambda) = \Sigma(-1)^q j(f_q;\lambda)$$

with, say,

$$j(f_q;\lambda) = \Sigma_{k=1}^{n_q} \frac{\lambda_{k,q}}{1-\lambda\lambda_{k,q}}, \qquad n_q = \pi(im\ f_q) \leq \pi(G_q);$$

the integers κ_k are then obtained by collecting equal summands. This concludes the proof.

<u>Lemma 3</u>. Let $f: G \to G$ be a homomorphism of a group sequence G, and let

$$gli(f;\lambda) \sim \Sigma_0^{\infty} J(f^{n+1})\lambda^n$$

be the formal power-series expansion as in [3, lemma 4]. If $\chi(im\ f) \neq 0$ then $J(f^s) \neq 0$ for some s with $1 \leq s \leq \varkappa(G)$.

(Proof.) From (1) there follows easily

$$J(f^s) = \Sigma_{k=1}^{n} \kappa_k \lambda_k^s,$$

with n, κ_k, λ_k as indicated there.
Now consider $J(f^s) = 0$, $1 \leq s \leq n$, as a system of linear equations in unknowns κ_k; the determinant Δ of the system is then readily recognised as

$$\Pi_1^m \lambda_k \times V(\ldots \lambda_k \ldots)$$

with V the Vandermonde determinant. Then $\Delta \neq 0$ since the λ_k's are distinct and non-zero; hence all $r_k = 0$ and in particular

$$0 = \Sigma_1^n r_k = -\chi\,(im f)\,.$$

This contradicts an assumption, and proves the assertion.

Remarks. Lemma 1 is obviously a result on the structure of the rational function gli ; it implies, e.g., that

$$\lim_{\lambda \to \infty} \lambda\, gli\,(f, \lambda) = -\chi\,(im f)\,,$$

interpreting the limit as $\frac{1}{\lambda}\, gli\,(f\,;\,\frac{1}{\lambda})$ at $\lambda = 0$. In particular, $gli\,(f\,;\lambda) \not\equiv 0$ if $\chi\,(im f) \neq 0$, so that some $J(f^s) \neq 0$; lemma 3 then gives more information concerning this integer s.

(Obviously the proof of lemma 2 is an improved version of that used in [3, corollary 2] and [4, theorem 2].) These two lemmas form the algebraic apparatus of the following theorem.

<u>Theorem 4</u>. Let X, Y be triangulable spaces, $\chi(Y) \neq 0$ and let f, g be continuous maps with

$$f: X \to Y\,, \qquad g: Y \to X\,,$$

(2) $\qquad f_*$ onto , $\quad ker\, g_* = 0$.

Then the map $gf: X \to X$ has some iterate $(gf)^s$ with a fixed point, and $1 \leq s \leq \varkappa\,(Y)$.

(Proof.) There is

$$im\,(gf)_* = g_*\,(im\, f_*) = im\, g_* \approx H_*(Y)$$

by assumption on f_*, g_* ; hence

$$\chi\,(im\,(gf)_*) = \chi\,(Y) \neq 0\,.$$

Our assertion then follows immediately from lemma 2 and the Hopf-Lefschetz theorem (applied to $(gf)^s$; or from [3, theorem 5] with $Y = \emptyset$).

Corollary 5. Let X be triangulable with $\chi(X) \neq 0$, and let $f: X \to X$ be homotopic (or homologuous) to a homeomorphism of X (or, more generally, assume that f_* is either $1-1$ or maps onto). Then some iterate f^s, $1 \leq s \leq \varkappa(X)$, has a fixed point.

(Proof: for the second map take the identity of X.)

Remark. Possibly it is not apparent that corollary to theorem 5 [3, p.28] is a special case of the preceding assertion. Indeed, let $X = S^{2n}$, so that $\chi(X) = \varkappa(X) = 2$; and let $f: S^{2n} \to S^{2n}$ be continuous. Now either $J(f) \neq 0$, and f has a fixed point by the Hopf-Lefschetz theorem. Or $J(f) = 0$; but then degree $f = -1$ and f_* is an isomorphism, so that, by corollary 5, f^2 has a fixed point.

There is an obvious obstacle to direct application of theorem 4 : it is difficult to verify conditions (2) (except for homeomorphisms, where this is trivially true; however, see the preceding remark). To illustrate, consider maps $E^1 \to S^1$. Evidently, there are even local homeomorphisms onto; however, no $f: E^1 \to S^1$ has f_* mapping onto, nor does any $g: S^1 \to E^1$ have $\ker g_* = 0$ (merely consider the homology groups). We shall now exhibit a class of maps satisfying (2).

Definition 6. Given a category, a morphism f is termed r-invertible if $ff' = 1$; a unit morphism, for some (associated) morphism f'; the dual concept is ℓ-invertibility.

Thus, if $ff' = 1$, then f is r-invertible and f' ℓ-invertible. As an example in the category of topological spaces, an inclusion map $Y \subset X$ is ℓ-invertible iff Y is a retract of X. Each invertible morphism is r- and

ℓ-invertible; converseely, an r - and ℓ-invertible morphism (or, more generally, an r-invertible monomorphism) is invertible. The composition of r-invertibles is r-invertible, so that, in particular, a morphism equivalent to an r-invertible is itself r-invertible.

From $ff' = 1$ it follows that f is epimorphic; more generally, for every admissible covariant functor F, $F(f)$ is r-invertible and hence epimorphic. In particular, on taking for F the homology functor,

Remark 7. In the category of triangulable spaces, if f is r-invertible and g ℓ-invertible, then f_* maps onto and $\ker g_* = 0$.

It is now seen that our invertibility conditions are rather brutal: we only need (2), but use a condition entirely independent of the structure of the homology functor. The following condition characterises r-invertible maps of compact topological spaces.

Lemma 8. Let $f: X \to Y$ be a continuous map of Hausdorff spaces. If f is r-invertible, there exists in X a closed section to the relation $fx = fy$; if X is compact, this latter condition is also sufficient.

(Proof.) Let $ff' = id_Y$ with $f': Y \to X$ continuous. Then $im f'$ is easily shown to be a section[1, p.78] to the relation $fx = fy$ in X; it is readily verified that $im f'$ is the set of fixed points of $ff': X \to X$, and hence closed if X is separated.

Conversely, let F be a compact section to the indicated relation; then one may prove directly that $f' = (f|F)^{-1}: Y \to X$ is single-valued and continuous, and obviously then $ff' = id_Y$.

Proposition 9. Let X, Y be triangulable, $\chi(Y) \neq 0$; let $f: X \to Y$ be r-invertible, $g: Y \to X$ ℓ-invertible. Then, for some s with $1 \leq s \leq \varkappa(Y)$, $(gf)^s$ has a fixed point.
(Proof: lemma 7 and theorem 4.)

Corollary 10. Let X, Y be triangulable, $\chi(Y) \neq 0$, and let $f, g: X \to Y$ be n-invertible; then there exist points $\{x_k\}_1^s$ in P, $1 \leq s \leq \varkappa(Y)$, such that

$$f x_k = g x_{k+1} \quad (1 \leq k < s), \ f x_s = g x_1 .$$

(Proof. Let $g g' = id_Y$; apply prop.9, obtaining a fixed point x_1 of $(g'f)^s$; define $x_{k+1} = g' f x_k$.)

In particular, for $s = 1$ there results a "coincidence theorem" as suggested in [4, p.91]:

Corollary 11. If $f, g: X \to Y$ are n-invertible, X, Y triangulable and Y homologically point-like (e.g. $Y = E^n$), then $fx = gx$ for some $x \in X$.

Corollary 12. If X is triangulable, $f: X \to Y$ n-invertible, with Y a retract of X and $\chi(Y) \neq 0$, then some iterate f^s has a fixed point, $1 \leq s \leq \varkappa(Y)$.
(Proof: apply prop. 9 with $g = j: Y \subset X$ the inclusion map, 1-invertible since Y is a retract; obviously $j(f(x)) = f(x)$.)

Remark 13. In assertions 9-12, the maps f, g may be replaced by homotopic (or homologuous) maps.

Further applications of theorem 4 will be given in a forthcoming paper on flows.

CORRECTIONS to preceding papers. The second displayed formula in corollary 1,[3,p.20], should end with ... = - rank G. In [3, p.29], 11th line from below, replace g:Y→Y by $g: X \to Y$.

In [4, p.89], 10th line from below, replace χ(T) by χ(X); two lines further down, the upper limit of summation should read χ(X) - 1 . On p.91, lines 2-3 from above, the sentence "If f itself ... holds" should be deleted completely.

References

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