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Homological fixed point theorems. III

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Theorem 2 of [4] asserts the presence of a fixed point for some one of the iterates \( f, f^2, \ldots, f^m \) of any continuous map \( f : X \to X \), under rather strict restrictions on \( X \) (non-oddness, l.c.; one may then take \( m = \chi(X) \)). In some cases it may be useful to weaken the conditions on \( X \) but restrict the maps \( f \) considered. Indeed, some theorems of this type are already known: assume \( X \) triangulable; if \( \chi(X) \neq 0 \) and \( f : X \to X \) is homotopic to the identity map, then \( f \) has a fixed point (a corollary to the Hopf-Lefschetz fixed point theorem); or, more generally, if \( f : X \to X \) is homotopic to a retraction \( X \to Y \) with \( \chi(Y) \neq 0 \), then again \( f \) has a fixed point (theorem 6 in [3]). The main result of this paper, theorem 4, is another result of this type. In particular, it is shown that if \( f : X \to X \) is homologous to a homeomorphism and \( \chi(X) \neq 0 \), then some iterate \( f^s \) has a fixed point (and an upper bound to \( s \) is given: corollary 5).

The terminology and notation of [3] are preserved. In particular, "group" means an abelian group \( G \) with fixed integrity domain \( J \) as left operators, and with finite rank over \( J \) (this rank will now be denoted by \( \tau(G) \)). A "group sequence" is a sequence \( \{ G_\alpha \} \) of such groups with \( \prod G_\alpha \) again of finite rank over \( J \). The Euler characteristic of \( G = \{ G_\alpha \} \) is, as in [3], defined as
\[ \chi(G) = \sum (-1)^{2r} \pi(G_r). \]

For \( n, j, q < \infty \) we refer to definitions 1 to 3 in [3]. It seems useful to introduce the following notation.

**Definition 1.** For a group sequence \( G = \{G_n\} \) set

\[ \delta(G) = \sum \pi(G_r). \]

Obviously \( \delta(G) = \pi(\prod G_r), \delta(G) \geq 0, \) and \( \delta(G) = 0 \) iff all \( G_n \) are periodic; \( \chi(G) \leq \delta(G), \) with equality iff all odd-indexed \( G_{2n+1} \) are periodic; \( \delta(G) \geq \chi(G) \) are both even integers.

For triangulable spaces \( X \) (i.e. topological spaces with a finite triangulation), the homology sequence is denoted by \( H_*(X) = \{H_n(X)\} \), the \( q \)-th Betti number by \( \beta_q(X) = \pi(H_q(X)); f_*: H_*(X_1) \to H_*(X_2) \) denotes the homomorphism induced by a continuous \( f: X_1 \to X_2 \); and we define

\[ \chi(X) = \chi(H_*(X)) = \sum (-1)^q \beta_q(X), \delta(X) = \delta(H_*(X)) = \sum \pi q(X). \]

In particular, \( \chi(X) = \delta(X) \) iff \( X \) is non-odd (cf. the definition in [4, p. 87]). (As another example, for compact 2-manifolds \( X, \delta(X) - \chi(X) = 4 \times (\text{genus of } X). \)

**Lemma 2.** If \( f: G \to G \) is a homomorphism of a group \( G, \) then

\[ j(f; \lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda}, \]

with \( n = \text{rank } \text{im } f = \pi(G), \) and \( 0 \neq \lambda \) in a root field over \( J. \) If \( f: G \to G \) is a homomorphism of a group sequence \( G, \) then

\[ qli(f; \lambda) = \sum_{n=1}^{\infty} \frac{\kappa_n \lambda^n}{1 - \lambda}, \]

with distinct \( \lambda \) and integers \( \kappa \) and

\[ \sum_{n=1}^{\infty} \kappa_n = -\chi(\text{im } f), \quad n \leq \delta(G). \]
(Proof.) In the definition of \( \varphi (\cdot) \) as
\[
\varphi (\cdot; A) = \det (I - \lambda D^{-1}A)
\]
onobviously
\[
\deg \varphi = \text{rank } D^{-1}A = \text{rank } A = \pi (\text{im } f).
\]
Hence the decomposition of \( \varphi (\cdot) \) in its root field \( J_\varphi \) over \( J \) may be written as (cf. proof of theorem 2 in [3]; note that \( \varphi (\cdot; 0) = \det I = 1 \))
\[
\varphi (\cdot; \lambda) = \prod_{\lambda_k \neq 0} (1 - \lambda \lambda_k)
\]
with \( 0 \neq \lambda_k \in J_\varphi \), \( \eta = \pi (\text{im } f) \). Then \( j = -\frac{d\varphi}{d\lambda} / \varphi \) yields the first assertion.

The second then results on applying the first to
\[
\text{gli} (\cdot; \lambda) = \sum (-1)^j j (\cdot; \lambda)
\]
with, say,
\[
j (f; \lambda) = \sum_{k=1}^{\eta} \frac{\lambda_k \quad \eta}{1 - \lambda \lambda_k}, \quad \eta = \pi (\text{im } f) \leq \pi (G);
\]
the integers \( \kappa_{\lambda_k} \) are then obtained by collecting equal summands. This concludes the proof.

**Lemma 3.** Let \( f : G \to G \) be a homomorphism of a group sequence \( G \), and let
\[
\text{gli} (f; \lambda) = \sum_0^\infty j (f, \lambda, \lambda^n)
\]
be the formal power-series expansion as in [3, lemma 4]. If \( \lambda (\text{im } f) \neq 0 \) then \( J (f^s) \neq 0 \) for some \( s \) with \( 1 \leq s \leq \sigma (G) \).

(Proof.) From (1) there follows easily
\[
J (f^s) = \sum_{\lambda_k} \kappa_{\lambda_k} \lambda_{\lambda_k}^s,
\]
with \( n, \kappa_{\lambda_k}, \lambda_{\lambda_k} \) as indicated there.

Now consider \( J (f^s) = 0, 1 \leq s \leq n \), as a system of linear equations in unknowns \( \kappa_{\lambda_k} \); the determinant \( \Delta \) of the system is then readily recognised as

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with $V$ the Vandermonde determinant. Then $\Delta \neq 0$ since the $\lambda_{ah}$'s are distinct and non-zero; hence all $\kappa_{ah} = 0$ and in particular

$$0 = \sum_{h} \kappa_{ah} = -\chi \left( \text{im} f \right).$$

This contradicts an assumption, and proves the assertion.

Remarks. Lemma 1 is obviously a result on the structure of the rational function $g(f); \lambda$; it implies, e.g., that

$$\lim_{\lambda \to \infty} \lambda g(f; \lambda) = -\chi \left( \text{im} f \right),$$

interpreting the limit as $\frac{1}{\lambda} g(f; \frac{1}{\lambda})$ at $\lambda = 0$.

In particular, $g(f; \lambda) \neq 0$ if $\chi \left( \text{im} f \right) = 0$, so that some $J(f^s) \neq 0$; lemma 3 then gives more information concerning this integer $s$.

(Obviously the proof of lemma 2 is an improved version of that used in [3, corollary 2] and [4, theorem 2].) These two lemmas form the algebraic apparatus of the following theorem.

**Theorem 4.** Let $X, Y$ be triangulable spaces, $\chi (Y) \neq 0$ and let $f, g$ be continuous maps with

$$f : X \to Y, \quad g : Y \to X,$$

(2) $f^{s}$ onto, hence $g^{s} = 0$.

Then the map $g f : X \to X$ has some iterate $(g f)^s$ with a fixed point, and $1 \leq s \leq \infty (Y)$.

(Proof.) There is

$$\text{im} (g f)^{s} = g_{X} (\text{im} f^{s}) = \text{im} g^{s} = H_{X} (Y)$$

by assumption on $f^{s}, g^{s}$; hence

$$\chi (\text{im} (g f)^{s}) = \chi (Y) \neq 0.$$

Our assertion then follows immediately from lemma 2 and the Hopf-Lefschetz theorem (applied to $(g f)^{s}$; or from [3, theorem 5] with $Y = \emptyset$).
Corollary 5. Let $X$ be triangulable with $\chi(X) \neq 0$, and let $f: X \to X$ be homotopic (or homologuous) to a homeomorphism of $X$ (or, more generally, assume that $f^X_*$ is either $1 - 1$ or maps onto). Then some iterate $f^s$, $1 \leq s \leq \infty$, has a fixed point.

(Proof: for the second map take the identity of $X$.)

Remark. Possibly it is not apparent that corollary to theorem 5 [3, p.28] is a special case of the preceding assertion. Indeed, let $X = S^{2n}$, so that $\chi(X) = \omega (X) = 2$; and let $f: S^{2n} \to S^{2n}$ be continuous. Now either $f^s(f) \neq 0$, and $f$ has a fixed point by the Hopf-Lefschetz theorem. Or $f^s(f) = 0$; but then degree $f = -1$ and $f^X_*$ is an isomorphism, so that, by corollary 5, $f^2$ has a fixed point.

There is an obvious obstacle to direct application of theorem 4: it is difficult to verify conditions (2) (except for homeomorphisms, where this is trivially true; however, see the preceding remark). To illustrate, consider maps $E^1 \to S^1$. Evidently, there are even local homeomorphisms onto; however, no $f: E^1 \to S^1$ has $f^X_*$ mapping onto, nor does any $g: S^1 \to E^1$ have $\ker g^X_*= 0$ (merely consider the homology groups). We shall now exhibit a class of maps satisfying (2).

Definition 6. Given a category, a morphism $f$ is termed $\kappa$-invertible if $ff' = 1$; a unit morphism, for some (associated) morphism $f'$; the dual concept is $\ell$-invertibility.

Thus, if $ff' = 1$, then $f$ is $\kappa$-invertible and $f'$ is $\ell$-invertible. As an example in the category of topological spaces, an inclusion map $Y \subset X$ is $\ell$-invertible iff $Y$ is a retract of $X$. Each invertible morphism is $\kappa$- and $\ell$-invertible.

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invertible; conversely, an \( \eta \) - and \( \ell \) -invertible morphism (or, more generally, an \( \eta \) -invertible monomorphism) is invertible. The composition of \( \eta \) -invertibles is \( \eta \) -invertible, so that, in particular, a morphism equivalent to an \( \eta \) -invertible is itself \( \eta \) -invertible.

From \( \ell \ell' = 1 \) it follows that \( \ell \) is epimorphic; more generally, for every admissible covariant functor \( F \), \( F(\ell) \) is \( \eta \) -invertible and hence epimorphic. In particular, on taking for \( F \) the homology functor,

**Remark 7.** In the category of triangulable spaces, if \( \ell \) is \( \eta \) -invertible and \( g \), \( \ell \) -invertible, then \( f_\ell \) maps onto and \( \ker g_\ell = 0 \).

It is now seen that our invertibility conditions are rather brutal: we only need (2), but use a condition entirely independent of the structure of the homology functor. The following condition characterises \( \eta \) -invertible maps of compact topological spaces.

**Lemma 8.** Let \( f : X \to Y \) be a continuous map of Hausdorff spaces. If \( f \) is \( \eta \) -invertible, there exists in \( X \) a closed section to the relation \( f x = f y \); if \( X \) is compact, this latter condition is also sufficient.

(Proof.) Let \( \ell \ell' = \text{id}_Y \) with \( \ell' : Y \to X \) continuous. Then \( \text{im} \ell' \) is easily shown to be a section[1, p.78] to the relation \( f x = f y \) in \( X \); it is readily verified that \( \text{im} \ell' \) is the set of fixed points of \( \ell \ell' : X \to X \), and hence closed if \( X \) is separated.

Conversely, let \( F \) be a compact section to the indicated relation; then one may prove directly that \( \ell' = (F(F))^{-1} \); \( Y \to X \) is single-valued and continuous, and obviously then \( \ell \ell' = \text{id}_Y \).

**Proposition 9.** Let \( X \), \( Y \) be triangulable, \( \chi(Y) \neq 0 \); let \( \ell : X \to Y \) be \( \eta \) -invertible, \( g : Y \to X \) \( \ell \) -invertible. Then, for some \( \ell \) with \( 1 \leq \ell \leq \chi(Y) \), \( (gf)\ell \) has a fixed point.

(Proof: lemma 7 and theorem 4.)

**Corollary 10.** Let \( X \), \( Y \) be triangulable, \( \chi(Y) \neq 0 \).

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and let $f, g : X \rightarrow Y$ be $\kappa$-invertible; then there exist points $\{x_{\kappa} \}_{i=1}^{s}$ in $P$, $1 \leq s \leq \text{ec}(Y)$, such that
\[ f(x_{\kappa}) = g(x_{\kappa+1}) \quad (1 \leq \kappa < s), \quad f(x_{s}) = g(x_{1}). \]

(Proof. Let $g = \text{id}_{Y}$; apply prop. 9, obtaining a fixed point $x_{1}$ of $(g^{f})^{s}$; define $x_{\kappa+1} = g^{f}x_{\kappa}$.)

In particular, for $s = 1$ there results a "coincidence theorem" as suggested in [4, p. 91]:

**Corollary 11.** If $f, g : X \rightarrow Y$ are $\kappa$-invertible, $X, Y$ triangulable and $Y$ homologically point-like (e.g. $Y = \mathbb{R}^m$), then $f(x) = g(x)$ for some $x \in X$.

**Corollary 12.** If $X$ is triangulable, $f : X \rightarrow Y$ $\kappa$-invertible, with $Y$ a retract of $X$ and $\chi(Y) \neq 0$, then some iterate $f^{s}$ has a fixed point, $1 \leq s \leq \kappa(Y)$.

(Proof: apply prop. 9 with $g = j : \gamma \subset X$ the inclusion map, $\kappa$-invertible since $Y$ is a retract; obviously $j(f(x)) = j(f(x))$.)

**Remark 13.** In assertions 9-12, the maps $f, g$ may be replaced by homotopic (or homologuous) maps.

Further applications of theorem 4 will be given in a forthcoming paper on flows.

CORRECTIONS to preceding papers. The second displayed formula in corollary 1, [3, p. 20], should end with $\ldots = - \text{rank } G$. In [3, p. 29], 11th line from below, replace $g:Y \rightarrow Y$ by $g:X \rightarrow Y$.

In [4, p. 89], 10th line from below, replace $\chi(T)$ by $\chi(X)$; two lines further down, the upper limit of summation should read $\chi(X) = 1$. On p. 91, lines 2-3 from above, the sentence "If $f$ itself ... holds" should be deleted completely.
References


