Essentially, this paper consists of the application of well-known fixed-point theorems, and others recently obtained in [4-6], to the existence problem of periodic solutions in abstract flows.

Section 1 gives the necessary definitions. The main part is section 2. Here there appear, first, two rather general theorems, 9 and 10; it will be apparent that theorem 9, in some form or other, is well-known; theorem 10 was suggested in [7]. The remaining theorems 11,12,15,16 treat more special situations, possibly not covered by the preceding results. Section 3 then contains only notes and remarks, and its latter part may serve as a link between flow theory and dynamical system theory. Its presence at the conclusion of the paper was dictated by the wish not to intersperse the preliminaries to section 2 with details not absolutely necessary.

For integral \( n > 0 \), \( \mathbb{R}^n \) denotes euclidean \( n \)-space, \( C^n \) its subset of points with non-negative integral coordinates, \( (E^1)^\infty \) the Hilbert parallelepiped, \( S^n \) the \( n \)-sphere, all with their natural topology; the first two are also taken with their natural additive structure and partial order.

\( P \) usually denotes a topological space; if triangulable, \( \pi_q(P) \) is its \( q \)-th Betti number, and \( \chi(P) = \sum (-1)^q \pi_q(P) \) its Euler characteristic. The composition of maps say \( f \) and \( g \) is usually denoted by \( f \circ g \), so that \( f \circ g(x) = f(g(x)) \).
1. FLOWS.

Convention 1. A semi-group shall mean a topological quasi-ordered semi-group (in the usual sense) with unit element. (Also see section 3.)

Constructions such as "the semi-group $R$" will be preferred to the more correct (but, for our purposes, unnecessarily pedantic) "the semi-group $(R, +, \geq, t)$" with $R$ a set and $+$, $\geq$, $t$ the semi-group, quasi-order and topology structures on $R$. In a similarly vague, but possibly obvious, sense we will say that a semi-group $R$ is, e.g., a group, or is discrete; if the maximal relation on $R$ is taken as the quasi-ordering (i.e. $\alpha \geq \beta$ always; this is indeed a quasi-order), then $R$ will be termed unordered. Typical examples: $R^\omega$, $C^\omega$, $R^n$ taken unordered. The unit of a semi-group $R$ is often denoted by $e$, elements of $R$ by lowercase Greek letters.

Definition 2. Let $P$ be a topological space, $R$ a semi-group. A semi-flow $T$ on $P$ over $R$ is a mapping with properties 1° - 3° listed below.

1° $T: \{[\alpha, \beta] \in R \times R : \alpha \geq \beta \} \times P \to P$ is continuous, in the induced topology. For fixed $\alpha \geq \beta$ in $R$, $T$ defines a continuous map $P \to P$, standardly denoted as $T_\beta^\alpha$; using this notation we require further that

$$2^\circ T_\alpha^\alpha = 1,$$

the identity map of $P$, for $\alpha \in R$,

$$3^\circ T_\beta^\alpha \circ T_\gamma^\beta = T_\gamma^\alpha$$

for $\alpha \geq \beta \geq \gamma$.

If $R$ is unordered, $T$ is called a flow.

Further terms: If $R$ is discrete, $T$ itself will be called discrete. If, for all $\alpha \geq \beta$, $\theta \geq e$,

$$\alpha + T_\theta^\beta + \theta = T_\theta^\beta,$$

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then the semi-flow $T$ will be termed stationary. For fixed $x \in P$, $T$ defines a continuous map $T_\sigma x : \{\alpha \in R : \alpha \geq \sigma\} \rightarrow P$, assuming the value $T_\sigma x$ at $\alpha = \sigma$; this map will be called the solution (of $T$) through $x$.

Remark. Probably it is evident that a semi-flow is, essentially, a special type of covariant functor. Thus, let $P, R$ be as in def. 3. Denote by $R^\wedge$ the category with objects $\alpha \in R$, morphisms $[\alpha, \beta] \in R \times R$ with $\alpha \geq \beta$, and composition

$$[\alpha, \beta][\beta, \gamma] = [\alpha, \gamma].$$

Let $P^\wedge$ be the category with $P$ as sole object, and continuous maps $P \rightarrow P$ as morphisms. Then a semi-flow $T$ on $P$ over $R$ defines a covariant $T^\wedge : R^\wedge \rightarrow P^\wedge$ by

$$T^\wedge[\alpha, \beta] = T_\beta;$$

conversely, a covariant functor $T^\wedge : R^\wedge \rightarrow P^\wedge$ similarly defines a discrete semi-flow on $P$ over $R$ (taken discrete).

Example 3. In a Banach space $P$, let

$$\frac{dx}{d\theta} = A(\theta)x \quad (x \in P, \theta \in R^1)$$

be a (homogeneous linear) differential equation with $A(\theta)$ a linear bounded operator $P \rightarrow P$ depending continuously on $\theta$.

For $\alpha, \beta$ in $R^1$ let $U(\alpha, \beta)$ be the corresponding resolvent operator [10, p.150]; then $T_\beta = U(\alpha, \beta)$ defines a flow on $P$ over $R^1$ (necessarily taken unordered).

Slightly more generally, let

$$\frac{dx}{d\theta} = f(x, \theta) \quad (x \in P, \theta \in R^1)$$

be a differential equation with $f : P \times R^1 \rightarrow P$ continuous, and with global existence and unicity of solutions, and con-
tinuous dependence of solutions on initial data (if \( P \) is finite-dimensional, the latter condition follows from the preceding). Take any \( x \in P, \alpha, \beta \in \mathbb{R}^1 \), and determine the unique solution \( y \) of (1) which satisfies \( y(\beta) = x \); then set
\[
\alpha / \beta x = y(\alpha).
\]
Obviously this defines a flow on \( P \) over \( \mathbb{R}^1 \); flows of this type may be termed differential. It may be noted that it is stationary iff \( f \) is independent of \( \theta \).

There are many other interesting and natural examples of flows, e.g. in ergodic theory (e.g. [9], or [2, chap. XVI]); (see also dynamical systems in section 3). However, example 3 is to be considered as the fundamental one for the purposes of the present paper.

**Lemma 4.** If \( T \) is a flow on \( P \) over \( \mathbb{R} \), then every \( \alpha / \beta \) is a homeomorphism \( P \approx P \) and
\[
\alpha / \beta \alpha / \beta^{-1} = \beta / \alpha.
\]
(Proof: \( \alpha / \beta \alpha / \beta^{-1} = \beta / \alpha = 1 \), \( \beta / \alpha \cdot \beta / \alpha = 1 \).)

**Definition 5.** Let \( T \) be a semi-flow on \( P \) over \( \mathbb{R} \), and assume given a \( \tau > \sigma \) in \( \mathbb{R} \). Then \( T \) is said to admit the period \( \tau \) if, for all \( \alpha > \sigma \),
\[
\alpha / \tau \sigma = \alpha / \tau \sigma \cdot \tau \sigma = \alpha / \tau \sigma.
\]

**Examples 6.** Every semi-flow admits the period \( \sigma \). A stationary semi-flow \( T \) admits all periods \( \tau > \sigma \);
\[
\alpha / \tau \sigma \cdot \tau \sigma = \alpha / \tau \sigma \cdot \tau \sigma = \alpha / \tau \sigma.
\]
As a partial converse, a flow admitting all periods is stationary: using lemma 4,
\[
\alpha / \beta = \alpha / \sigma \cdot (\beta / \sigma)^{-1}
\]
for all \( \alpha, \beta \), so that

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A differential flow (cf. (1), example 3) admits a period \( \tau \geq \sigma \) iff, for each fixed \( x \in \mathcal{P} \), \( f(x, \theta) \) has period \( \tau \) in \( \Theta \).

(This may be proved by showing that the latter condition is equivalent to: \( \gamma (\theta + \tau) \) is a solution of (1) whenever \( \gamma (\theta) \) is.) In the first case of example 3 this is, of course, the familiar Floquet's theorem (e.g. [12, III, § 2]).

Lemma 7. Let \( \mathcal{T} \) be a semi-flow on \( \mathcal{P} \) over \( \mathbb{R} \), admitting a period \( \tau \geq \sigma \). Then \( \mathcal{T} \) also admits all periods \( n \tau \), \( n > 0 \) integral, and

\[
(3) \quad n \tau \sigma = \tau \sigma .
\]

If \( \mathcal{R} \) is an unordered (topological) group, then this holds for all integers \( n \) without restriction.

(Proof.) Using (2) thrice one obtains

\[
\alpha \cdot 2 \tau \sigma = (\alpha + \tau) \tau \sigma = \alpha \tau \tau \sigma = \alpha \tau \sigma = \alpha \sigma = \tau \sigma = \tau \sigma = \tau \sigma = \tau \sigma ,
\]

and by induction,

\[
(4) \quad \alpha + n \tau \sigma = \alpha \tau \sigma \quad \text{for} \quad n > 0 .
\]

Hence, for \( \alpha = \tau \),

\[
(m+1) \tau \sigma = \tau \sigma \quad \text{for} \quad (m+1) > 0 .
\]

so that, by induction, \( n \tau \sigma = \tau \sigma \quad \text{with} \quad (4) \) this completes the proof of the first statement.

Finally, if \( \mathcal{R} \) is an unordered group, then from (4)

\[
(5) \quad \alpha \tau \sigma = (\alpha - n \tau) + n \tau \sigma = \alpha - n \tau \sigma + n \tau \sigma ,
\]

and in particular \( -\tau \sigma = \tau \sigma - \tau \sigma \quad \text{(cf. lemma 4)}.\) Thus from (5),

\[
\alpha - n \tau \sigma = \alpha \tau \sigma = \tau \sigma - n \tau \sigma ,
\]

as was to be shown.

Remark. It may be remarked that for flows, property (3) is...
equivalent with stationarity of the "sampled" flow on $P$ over $\mathcal{C}^1$, defined by
$$m \tau T_m \tau \quad (m \geq n \text{ in } \mathcal{C}^1).$$

2. PERIODIC SOLUTIONS.

Throughout this section $P$ denotes a topological space and $R$ a semi-group (cf. convention 1 and section 3).

As may be expected, a solution $T_\tau x$ is called $\tau$-periodic (a semi-flow on $P$ over $R$, $x \in P$, $\tau > \sigma$) if
$$T_{\theta + \tau} x = T_\tau x \quad \text{for all } \theta \geq \sigma.$$
(This is current usage if $R = R^1$ is taken unordered; if $R = R^1$ with natural order, the term is so used in Laplace transform theory.) Obviously, a $\tau$-periodic solution is $n\tau$-periodic for all integers $n \geq 0$.

The main tool used to obtain conditions for existence of periodic solutions is the following

**Proposition 8.** Let $T$ be a semi-flow on $P$ over $R$ admitting a period $\tau \geq \sigma$. For $x \in P$, the solution $T_\tau x$ is $\tau$-periodic iff $x$ is a fixed point of $T_\tau : P \to P$.

(Proof.) This is direct verification. If $T_\tau x$ is $\tau$-periodic, then $T_{\theta + \tau \sigma} x = T_\tau x \quad \text{for all } \theta \geq \sigma$; for $\theta = \sigma$ one obtains $T_\tau x = x$, a fixed point of $T_\tau$.

Conversely, if $T_\tau x = x$, then
$$T_{\theta + \tau \sigma} x = T_\tau x \quad \text{for all } \theta \geq \sigma,$$
i.e., $T_\tau x$ is $\tau$-periodic.

Proposition 8 will be applied, without further reference, in reading off existence of periodic solutions from various fixed-point theorems. In each pair of theorems 9-10, 11-12, 15-16 there appear similar results under varied assumptions
Theorem 9. Let \( T \) be a semi-flow on \( P \) over \( R \), admitting a period \( \tau > \sigma \). If there exists an \( X \subset P \) which is a retract of \( (E^4)^\infty \) and has \( T_\sigma^\infty X \subset X \), then there exists a \( \tau \)-periodic solution.

(Proof.) Partialised \( T_\sigma^\infty : X \to X \) is continuous; apply the Schauder-Tichonov fixed-point theorem [11, p.263].

Note that the conclusion obtains, in particular, if \( P \) itself is a retract of \( (E^4)^\infty \).

Theorem 10. Let \( T \) be a semi-flow on \( P \) over \( R \), admitting a period \( \tau > 0 \); assume that \( P \) is triangulable with \( \chi(P) \neq 0 \), and that \( \{ \theta \in R : \theta > \sigma \} \) is connected.

(Proof.) Denote by \( J(f) \) the Lefschetz invariant of a continuous map \( f : P \to P \) (cf.[1, p.598], or [4]). By assumption, \( \tau_\sigma^{\infty} \) depends continuously on \( \theta > \sigma \); from [4, lemma 7] it then follows that \( J(\tau_\sigma) \) also varies continuously with \( \theta \). Since \( J(f) \) is integer-valued and \( \{ \theta \in R : \theta > \sigma \} \) connected, \( J(\tau_\sigma) \) is constant. Therefore

\[
J(\tau_\sigma) = J(\tau_\sigma^\infty) = J(1) = \chi(P) \neq 0 .
\]

By the Lefschetz-Hopf fixed-point theorem, there exists a \( \tau \)-periodic solution of \( T \).

Remark. Theorem 10 applies a fortiori if \( R \) is arcwise connected, e.g. for \( R = R^1 \). In this case the proof may be simplified, omitting all reference to [4] and lemma 17, as follows: use the assumed path from \( \sigma \) to \( \tau \) in \( R \) to show that \( \tau_\sigma = 1 \) is homotopic to \( \tau_\sigma^\infty \); then again \( J(\tau_\sigma^\infty) = J(1) \neq 0 \). This was the idea of [7, theorem].

Theorem 11. Let \( T \) be a flow on \( P \) over \( R \), admitting a period \( \tau > \sigma \); and assume that \( P \) is triangulable with...
1 \leq n \leq \sum \pi_q(P).

(Proof.) From lemma 4, \( T^\sigma \) is now a homeomorphism \( P \approx P \); apply corollary 5 of [6].

**Theorem 12.** Let \( T \) be a semi-flow on \( P \) over \( \mathbb{R} \), admitting a period \( \tau > 0 \); assume that \( P \neq \emptyset \) is non-odd. Then there exists an \( n \tau \)-periodic solution with \( 1 \leq n \leq \sum \pi_q(P) = \chi(P) \).

(Proof: [5, theorem 2].)

**Remark.** Non-oddness is a concept introduced in [5, definition 7]: \( P \) is non-odd if \( \pi_{2q+1}(P) = 0 \) for all \( q \), i.e. if all odd-dimensional homology groups are periodic. In particular, then, each semi-flow on \( S^{2n} \) admitting a period \( \tau > 0 \) has a \( 2\tau \)-periodic solution.

Before presenting the next two theorems, it will be necessary to introduce and illustrate another concept. A continuous map \( F: P \to P \) will be termed a symmetry of \( P \) if \( F^2 = 1 \); necessarily, then, \( F: P \approx P \) homeomorphically.

**Definition 13.** Let \( F \) be a symmetry of \( P \), and \( T \) a semi-flow on \( P \) over \( \mathbb{R} \). Then \( T \) will be termed \( F \)-symmetric if each \( T^\sigma \) commutes with \( F \).

**Example 14.** Let \( T \) be a differential flow on a Banach space \( P \) over \( \mathbb{R}^l \), defined by a differential equation (1) as in example 3. Also, let \( F \) be a linear symmetry of \( P \). Then \( T \) is \( F \)-symmetric iff \( Ff(x, \theta) = f(Fx, \theta) \) for all \( x \in P \), \( \theta \in \mathbb{R}^l \) (hint: show that \( Fy \) is a solution of (1) iff \( y \) is). E.g. the flow described by \( dx/d\theta = A(\theta)x \) is \( F \)-symmetric for \( F \) defined by \( Fx = -x \).

Physical systems with \( n \) degrees of freedom are often described by differential equations such as
These may be "reduced" to systems of type (1) by a familiar device,

\[
\frac{d^2 x}{d \theta^2} = f(x, \frac{dx}{d \theta}, \theta) \quad (x \in \mathbb{R}^n, \theta \in \mathbb{R}^1).
\]

with \([x, \mu] \in \mathbb{R}^{2n}\). If, as sometimes happen,

\[
f(-x, \mu, \theta) = -f(x, \mu, \theta), \quad ([x, \mu, \theta] \in \mathbb{R}^{2n+1})
\]

then (under the appropriate conditions on \(f\)) (6) defines a flow on \(\mathbb{R}^{2n}\) over \(\mathbb{R}^1\); this flow is then \(F\)-symmetric for \(F\) defined by

\[
F[x, \mu] = [-x, \mu].
\]

**Theorem 15.** Let \(F\) be a symmetry of \(P\), and \(T\) an \(F\)-symmetric semi-flow on \(P\) over \(R\), admitting a period \(\tau > \sigma\). If there exists a subset \(X \subset P\) with \(X\) a retract of \((E^1)^\infty\) and

\[
\tau \sigma X = FX,
\]

then there exists a \(2\tau\)-periodic solution of \(T\).

(Proof.) Recall that \(F = F^{-1}\). Partialised \(F \circ \tau^\sigma: X \rightarrow X\), so that (Schauder-Tichonov) there is a fixed point \(x\) of \(F \circ \tau^\sigma\). Then also \(\tau^\sigma x = FX\), and, using lemma 7 and \(F\)-symmetry,

\[
2\tau^\sigma x = \tau^\sigma x = \tau^\sigma x = \tau^\sigma x = \tau^\sigma x = x.
\]

is a fixed point of \(\tau^\sigma\).

Remarks. This is an abstract form of the Poincaré symmetry principle for dynamical systems in \(R^2\) [12, p.145]. Obviously (7) is satisfied if \(X = \sigma\), i.e. if \(P\) itself is a retract of \((E^1)^\infty\).

**Theorem 16.** Let \(F\) be a symmetry of \(S^{2n}\), \(T\) an
$F$-symmetric semi-flow on $S^2$ over $\mathbb{R}$, admitting a period $\tau > \sigma$. If $F$ has no fixed-point, then $T$ has at least two $2\tau$-periodic solutions.

(Proof.) From theorem 11, $T$ has at least one $2\tau$-periodic solution. These are in 1-1 correspondence with the fixed points $x$ of $T$. From $F$-symmetry there then follows $T^\tau F x = F x$, so that $F x = x$ if there is only one $2\tau$-periodic solution. This contradicts the assumption on $F$ and concludes the proof.

Remarks. The assertion may also be formulated thus: either there is at least one non-constant $2\tau$-periodic solution, or there are at least two constant solutions. In the case that $F$ is a negative symmetry (i.e. degree $F = -1$), the existence of one $2\tau$-periodic solution also follows from [5, theorem 3].

3. ADDENDA.

For definiteness in convention 1, a semi-group means some $(\mathbb{R}, +, \geq, t)$ where $\mathbb{R}$ is a set and $+, \geq, t$ are structures as follows.

The $+$ is a semi-group operator, i.e. a binary associative operator on $\mathbb{R}$; there exists a unit $\sigma \in \mathbb{R}$ ($\alpha + \sigma = \sigma + \alpha = \alpha$ always). For integral $m > 0$ and $\alpha \in \mathbb{R}$ we write $m \alpha = \alpha + \alpha + \cdots + \alpha$ ($n$ terms), $0\alpha = \sigma$.

The $\geq$ is a quasi-order in $\mathbb{R}$, i.e. a reflexive and transitive relation (laxly speaking, a partial ordering less the anti-symmetry condition [2, I, § 4]). The advantage is that a single formulation serves for both the interesting cases, of $\geq$ a
partial order, and also of the maximal relation on $\mathbb{R}$ ($\alpha \geq \beta$ always); in the latter case the semi-group was termed unordered. Lastly, $t$ is a topology on $\mathbb{R}$.

We require, further, these compatibility conditions:

(i) $\alpha \geq \beta$ and $\alpha' \geq \beta'$ implies $\alpha + \alpha' \geq \beta + \beta'$;

(ii) $+$ is continuous, considered as a map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (in the induced topology);

(iii) the set $\{ [\alpha, \beta] : \alpha \geq \beta \}$ is closed in $\mathbb{R} \times \mathbb{R}$.

Since exchange of coordinates is a homeomorphism of $\mathbb{R} \times \mathbb{R}$, $\{ [\alpha, \beta] : \beta \geq \alpha \}$ is also closed.

**Lemma 17.** Let $\mathbb{R}$ be a partially ordered semi-group. Then

1° $\mathbb{R}$ is a Hausdorff space,

2° if $\mathbb{R}$ is connected and $\{ \theta : \theta > \sigma \}$ open, then $\mathbb{R}^+ = \{ \theta : \theta \geq \sigma \}$ is connected;

3° for $\alpha \geq \sigma$ the set $\{ n\alpha \}_{n \in \mathbb{Z}^+}$ is discrete.

(Proof.) The diagonal in $\mathbb{R} \times \mathbb{R}$ is the intersection of closed sets

$\{ [\alpha, \beta] : \alpha \geq \beta \}$, $\{ [\alpha, \beta] : \beta \geq \alpha \}$,

and hence is also closed. Thus the Bourbaki condition is satisfied and one has 1° (cf. theorem 13 in [2, chap.IV]).

Next, assume $\mathbb{R}^+$ is not connected. Since it is closed, as a section of $\{ [\alpha, \beta] : \alpha \geq \beta \}$ over $\sigma$, there exists a non-trivial decomposition into closed sets,

$\mathbb{R}^+ = A \cup B$, $\sigma \in B$.

Set $C = \mathbb{R} - \mathbb{R}^+$, so that one has the decomposition

$\mathbb{R} = A \cup (B \cup C)$.

As $\mathbb{R}$ is connected, to obtain a contradiction it suffices to show that $A \cap C = \emptyset$. Assume $\gamma_i \in C$, $\gamma_i \to \gamma \in A$. Since
\( \sigma \in B \), \( \gamma > 0 \) and hence is in the open set \( \{ \theta : \theta > \sigma \} \); then \( y_i > \sigma \) for some \( i \), contradicting \( y_i \in C \subset R - R_+ \).

This proves 2°.

For 3°, assume \( h_n \to + \infty \), \( h_n \alpha \to n \alpha \) with \( h_n \in \mathbb{N} \), \( n \in C^+ \), \( \alpha > \sigma \). Take any \( s > \alpha \); then \( h_n \alpha > s \alpha \) for large \( n \), and hence
\[
\alpha \leftarrow h_n \alpha > s \alpha > n \alpha .
\]
Therefore \( s \alpha = n \alpha \) for all \( s > \alpha \), and \( \{ n \alpha \}_{n \in C^+} \) is discrete.

**Definition 18.** Let \( P \) be a topological space, \( R \) a semi-group. A continuous map \( \tau : P \times \{ \theta \in R : \theta \geq \sigma \} \to P \) (to be written as a binary operator) with the properties
\[
x \tau \sigma = x, \quad (x \tau \theta) \tau \theta' = x \tau (\theta' + \theta)
\]
(for all \( x \in P \), \( \theta > \sigma \leq \theta' \)) is called a semi-dynamical system on \( P \) over \( R \); and, if \( R \) is unordered, a dynamical system on \( P \) over \( R \). (For the case \( R = R^1 \) see "unilateral" in [7], and "global semi-dynamical" in [8].)

**Lemma 19.** A stationary (semi-)flow \( \tau \) defines a (semi-) dynamical system \( \tau \) (both on \( P \) over \( R \)) by
\[
x \tau \theta = \tau \alpha \tau \beta \quad \text{for} \quad x \in P, \quad \theta > \sigma .
\]
If \( R \) is a group then every (semi-) dynamical system \( \tau \) defines a stationary (semi-) flow \( \tau \), both on \( P \) over \( R \), by
\[
\tau \alpha x = x \tau (\alpha - \beta) \quad \text{for} \quad x \in P, \quad \alpha > \beta .
\]
(Proof: direct verification).

On passing to a different space, even non-stationary flows may be described in terms of dynamical systems:

**Lemma 20.** If \( \tau \) is a (semi-) flow on \( P \) over \( R \), then
\[
\{ x, \alpha \} \tau \theta = \{ \theta + \alpha \tau x, \theta + \alpha \}
\]
(\( x \in P, \alpha \in R, \theta > \sigma \) defines a (semi-) dynamical system \( \tau \).
on $P \times R$ over $R$; the solution $T \times \theta$ is then the projection of $[x, \sigma] \times \theta$.

(Proof: direct verification)

In this connection, $P$ is sometimes called the phase space of $T$, and $P \times R$ its solution space. The semi-dynamical system defined by $(\theta)$ is somewhat singular; thus, if $R = R^1$ then there are no critical points nor cycles (in fact, $P \times (0)$ is a section generating $P \times R^1$).

References