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PROJECTIVELY GENERATED CONTINUITY STRUCTURES: A CORRECTION.

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The author wishes to state that an error occurs in his note "On certain projectively generated continuity structures", Celebrazioni archimedee del secolo XX, Simposio di topologia, 1964; pp. 47-50 (referred to as [PG] in what follows). This error affects the validity of assertions concerning "Case (1)", i.e., the case where a compact topological space X and the module, denoted by Φ, of all continuous functions on X are considered.

In [PG], the following Λ-structure μ_Φ on X has been introduced (by definition, a Λ-structure on a set X is a locally convex topology on the module ΛX of all finite formal linear combinations $\sum \lambda_i x_i$ of elements $x_i \in X$): μ_Φ is the finest locally convex topology on ΛX for which every continuous linear form coincides with the linear extension Λf of some continuous function f on X. Now, assertions made about μ_Φ in [PG] should refer to another structure ν, described below; the outline of a proof of an assertion on μ_Φ given in [PG] concerns, in fact, the structure ν instead of μ_Φ.

We are now going to state and prove propositions concerning ν.

<u>Proposition</u>. Let X be a completely regular separated

topological space. Consider the topology ν on ΛX generated by all mappings $\Lambda T : \Lambda X \to E$, where T is a continuous mapping of X into a locally convex topological linear (abbreviated l.c.t.l.) space E and ΛT is the linear extension of T. Then ν is the finest locally convex topology on ΛX under which the natural embedding (assigning $1 . x \in \Lambda X$ to $x \in X$) of X into ΛX is continuous. If F is a continuous mapping of X into a l.c.t.l. space E, then its linear extension $\Lambda F : (\Lambda X, \nu) \to E$ is also continuous; if this condition holds for a l.c.t.l. space $(\Lambda X, \nu')$ and the natural embedding of X into ΛX is continuous, then $\nu' = \nu$.

Definition. The space $(\Lambda X, \nu)$ (or any space isomorphic to it) is said to be freely generated by the topolcgical space X.

Proof of Proposition. It is clear that ν is a locally convex topology and that the natural embedding $\varepsilon : X \to \Lambda X$ is continuous (in fact, a homeomorphism). If $(\Lambda X, \nu')$ is a l.c.t.l. space and ε is continuous, then $\Lambda \varepsilon : \Lambda X \to \Lambda X$ is one of the mappings ΛT involved in the definition of ν; thus, ν is finer than ν'. The second assertion is obvious, for ΛF is one of the mappings generating ν. Let now the condition in question hold for $(\Lambda X, \nu')$. Since $\varepsilon : X \to$ $\to (\Lambda X, \nu)$ is continuous, so is $\Lambda \varepsilon : (\Lambda X, \nu') \to$ $\to (\Lambda X, \nu)$; hence ν' is finer than ν and, similarly, ν is finer than ν'.

Conventions. If $\mathcal{X}$ is a l.c.t.l. space, we denote by $\pi \mathcal{X}$ its completion. - If X is a compact topological space, and f is a continuous function on X, then we denote by $\tilde{f}$ the continuous linear extension of f to $\pi(\Lambda X, \nu)$.

__Theorem__. Let X be a compact topological space. There exists exactly one bijective linear mapping of $\pi(\Lambda X, \nu)$, the completion of the linear space freely generated by X, onto $C(X)'$ such that, denoting by σ_ξ the element of $C(X)'$ assigned to ξ, we have $\tilde{f}(\xi) = \sigma_\xi(f)$ for any $\xi \in \pi(\Lambda X, \nu)$ and any $f \in C(X)$.

__Proof__. I. Let $\xi \in \pi(\Lambda X, \nu)$; denote by σ_ξ the mapping which assigns $\tilde{f}(\xi)$ to $f \in C(X)$. We are going to show that $\sigma_\xi : C(X) \to R$ is continuous. Suppose the contrary; then there exist $f_n \in C(X)$, $n = 1,2,\ldots,$ such that $|f_n| \to 0$, $\tilde{f}_n(\xi) = 1$. For every $x \in X$, we have $\{f_n(x)\} \in (c_0)$; denote by F the mapping of X into (c_0) assigning $\{f_n(x)\}$ to x. It is easy to see that F is continuous linear. Denote by G the continuous linear extension of F to a mapping of $\pi(\Lambda X, \nu)$ into (c_0). Then G, restricted to ΛX, is one of the mappings generating ν; therefore, for any $h \in (c_0)'$, $h \circ G$ is a continuous linear form on $\pi(\Lambda X, \nu)$. Let $h_n \in (c_0)'$ assign α_n to $\{\alpha_k\} \in (c_0)$. Then, clearly, $h_n(G(z)) = \tilde{f}_n(z)$ for every $z \in \Lambda X$. From this it follows, by continuity, that $h_n(G(\xi)) = \tilde{f}_n(\xi)$. Since $\tilde{f}_n(\xi) = 1$, we obtain $h_n(G(\xi)) = 1$, $n = 1,2,\ldots,$ which is impossible, for $G(\xi) \in (c_0)$. This contradiction proves that σ_ξ is a continuous linear form on $C(X)$, that is, $\sigma_\xi \in C(X)'$.

II. Clearly, every continuous linear form g on $\pi(\Lambda X, \nu)$ is equal to some $\tilde{f}$ with $f \in C(X)$. Thus, if $\xi \in \pi(\Lambda X, \nu)$, $\xi \neq 0$, then $\tilde{f}(\xi) \neq 0$ for some $f \in C(X)$. This proves that the mapping which assigns σ_ξ to

ξ is one-to-one. - III. It remains to prove that, given an element $\tau \in C(X)'$, there exists a $\xi \in \pi(\Lambda X, \nu)$ with $\sigma_\xi = \tau$. Let $\bar{\tau}$ denote the measure on X corresponding to τ (that is, we have $\int f \, d\bar{\tau} = \tau(f)$ for every $f \in$ $\in C(X)$). Well known properties of integrals imply that if F is a continuous mapping of X into a Banach space E, then, for every $\varepsilon > 0$, there exists a point $z \in \Lambda X$, $z = \sum \lambda_i x_i$, such that $|\int F \, d\bar{\tau} - \sum \lambda_i F(x_i)| < \varepsilon$. For any continuous mapping F of X into a Banach space and any $\varepsilon > 0$, let now $U_{F,\varepsilon}$ denote the set of those points $z = \sum \lambda_i x_i \in \Lambda X$ for which $|\int F \, d\bar{\tau} -$ $- \sum \lambda_i F(x_i)| < \varepsilon$.

Since $U_{F,\varepsilon}$ are non-void, it is clear that $U_{F,\varepsilon}$ form a base of a filter $\mathscr{F}$. As it is easy to see, the mappings F of the just described kind generate the structure ν. Therefore, the filter $\mathscr{F}$ is a Cauchy filter. Let ξ be its limit point. It is easy to prove that $\sigma_\xi = \tau$.

Remark. It is evident that the original topology of X is induced by ν. As for the structure μ_Φ, $\Phi = C(X)$, introduced in [PG], the situation is quite different, as shown by simple well-known examples. Let, e.g., H be the Hilbert space of sequences $\{\xi_n\}$, $\sum |\xi_n|^2 < \infty$. Let H_w denote the same space endowed with the weak topology, and let $X \subset H_w$ consist of 0 and the points $e_n = \{\delta_{nk}\}$, where $\delta_{nk} = 1$ for $k = n$, $\delta_{nk} = 0$ for $k \neq n$. Then, clearly, $x_n \to 0$ and x_n are isolated in X. Consider the identity mapping $J : X \to H$. It is easy to see that the linear extension of J to $\tilde{J} : (\Lambda X, \mu_\Phi) \rightarrow H$ is continuous; thus μ_Φ induces the discrete topology on X.

The structure μ_Φ, $\Phi = C(X)$, seems to possess some interesting properties. We intend to return to these elsewhere.