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DETERMINATION OF EIGENVALUES AND EIGENFUNCTIONS OF BOUNDED SELF-ADJOINT OPERATORS

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(Preliminary communication)

1. The problem of determining the eigenvalues and eigenfunctions of self-adjoint bounded operators has been developed by many authors. L.V. Kantorovič [1] used the method of steepest descent to determine the largest eigenvalue, and the corresponding eigenfunction, of completely continuous self-adjoint positive definite operators in Hilbert space. Later M.A. Krasnoselskij [2] suggested ten methods for calculation of eigenvalues in n-dimensional spaces, but without proofs. These and Kostarčuk's [3] methods are simpler in comparison with [1]. The fifth method from [2] was investigated by B.P. Pugačev [4] under the assumption that the linear bounded operator is self-adjoint and positive definite. Wang Jin-ru [5] improved the fourth method from [2] and performed a comparison of some these gradient methods. Another method was proposed by W. Karush [6] .

In this note we shall deal with two methods which were described in [7],[8] . We assume throughout that H is a real Hilbert space. The basic idea of these methods is the following. Let us consider the equation

$$Ax - \lambda x = 0 , \tag{1}$$

where A is linear bounded operator in H, λ is a real parameter. Suppose that A is a positive self-adjoint operator in H (A is said to be positive if $(Ax,x) > 0$ for every $x \in H$, $x \neq 0$). We solve (1) by an iterative process

$$x_{n+1} = \frac{1}{\lambda_{n+1}} Ax_n , \tag{2}$$

where the parameters λ_n ($n = 1,2,\dots$) are to be determined from the condition that the functional $\| Ax - \tau x\|^2$ for the given element $x = x_n \in H$ is to assume its minimum on the set $\mathcal{R}$ ($\tau \in \mathcal{R}$) of all real numbers. Let us denote that value τ (dependent on n) by λ_{n+1}. Then we obtain that

$$\lambda_{n+1} = \frac{(Ax_n , x_n)}{\| x_n \|^2} \tag{3}$$

and

$$x_{n+1} = \frac{\| x_n \|^2}{(Ax_n , x_n)} Ax_n , \quad x_0 \neq 0 , \quad x_0 \in H \quad (n = 0,1,2,\dots). \tag{4}$$

The second method was proposed by I.A. Birger [9] but without any assumptions or convergence proofs. His method is as follows: Let

$$y_{n+1} = \mu_{n+1} Ay_n , \quad \mu_{n+1} = \frac{(Ay_n , y_n)}{\| Ay_n \|^2}, \quad y_0 \neq 0 , \quad y_0 \in H . \tag{5}$$

<u>Theorem 1</u> ([7],[8]). Let A be a non-negative [$(Ax,x) \geq 0$ for every $x \in H$] completely continuous self-adjoint operator in H, let N be the null set of A and let $x_0 \in H \ominus N$, $y_0 \in H \ominus N$ be not orthogonal to the eigenspace $H_{\lambda_1^*}$ corresponding to the first eigenvalue λ_1^*

of A . Then $\{\lambda_n\}$ is monotone increasing and converges to λ_1^* . The sequence $\{\mu_n\}$ is monotone decreasing and converges to $(\lambda_1^*)^{-1}$. Both sequences $\{x_n\}$, $\{y_n\}$ converge in $H \ominus N$ to one of the eigenfunctions corresponding to λ_1^* .

These methods were generalized by I. Marek [10] for linear bounded operators in Banach space, which have a dominant eigenvalue. Simultaneously with [1], the method (5) was investigated by H.F. Bückner [11]. The purpose of this note is to show that the sequences $\{\lambda_n\}$, $\{\mu_n\}$ also converge in the case when the greatest point of the spectrum $\sigma(A)$ of A is not an eigenvalue of A , to remove the condition that λ_1^* be an isolated point of $\sigma(A)$ and to give some estimates. The proofs are omitted and will be published later, together with further theorems.

2. Suppose that A is linear self-adjoint positive operator in H . Let $\tilde{\lambda}_1$ be the greatest element and m the smallest element of the spectrum $\sigma(A)$. The spectrum $\sigma(A)$ lies in the segment $\langle m, \tilde{\lambda}_1 \rangle$, where $m = \inf_{\|x\|=1} (Ax,x)$, $\tilde{\lambda}_1 = \sup_{\|x\|=1} (Ax,x)$, $m \geq 0$. (The class of self-adjoint positive definite operators $(m > 0)$ is encluded in the class considered here.) Let $\{E_\lambda\}$ be the spectral family of A .

<u>Theorem 2</u>. Let A be a self-adjoint positive operator in H . Suppose $E_\lambda x_0 \neq x_0$ for $\lambda < \tilde{\lambda}_1$, (or that $E_\lambda y_0 \neq y_0$ for $\lambda < \tilde{\lambda}_1$,). Then $\{\lambda_n\}$ is monotone increasing and converges to $\tilde{\lambda}_1$ (and $\{\mu_n\}$ is monotone decreasing and converges to $\tilde{\lambda}_1^{-1}$).

Theorem 3. Under the assumptions of Theorem 2 let $\tilde{\lambda}_1$ not be an eigenvalue of A. Then both $\{x_n\}$, $\{y_n\}$ converge to $\emptyset$ weakly in H.

Theorem 4. Let A be a positive self-adjoint operator in H and suppose that $\tilde{\lambda}_1$ (not necessarily and isolated point of $\sigma(A)$) is an eigenvalue of A, $H_{\tilde{\lambda}_1}$ is the eigenspace corresponding to $\tilde{\lambda}_1$ and that the projection of x_0 ($x_0 \in H$) on $H_{\tilde{\lambda}_1}$ is $\xi_1^{(0)} e_1$, where $e_1 \in H_{\tilde{\lambda}_1}$, $\| e_1 \| = 1$, $\xi_1^{(0)} > 0$. Then

$\lim\limits_{k \to \infty} \| x_k - N e_1 \| = 0$, where $N = \sup\limits_{k=1,2,\dots} \| x_k \| < +\infty$.

Now set $\cos(x,y) = \dfrac{(x,y)}{\|x\| \, \|y\|}$, $\sin(x,y) = \sqrt{1 - \cos^2(x,y)}$.

Then the following theorem holds.

Theorem 5. Let A be a positive self-adjoint operator in H, $E_\lambda x_0 \neq x_0$ for $\lambda < \tilde{\lambda}_1$, and suppose $\tilde{\lambda}_1$ is an isolated point of $\sigma(A)$. Then there exists a real q, $0 < q < 1$ such that for n_0 sufficiently large,

$$\tilde{\lambda}_1 - \frac{(Ax_{n_0+p}, x_{n_0+p})}{\| x_{n_0+p} \|^2} \leq q^{2p} \left(\tilde{\lambda}_1 - \frac{(Ax_{n_0}, x_{n_0})}{\| x_{n_0} \|^2} \right),$$

$$\| x_{n_0+p} - e_1 \| x_{n_0+p} \| \| \leq \sqrt{2}\, q^p [\| x_{n_0+p} \| (\|x_{n_0}\| - (x_{n_0}, e_1))]^{1/2}$$

$$\sin(x_{n_0+p}, e_1) \leq \frac{q^p}{(\tilde{\lambda}_1 - M)^{1/2}} (\tilde{\lambda}_1 - \lambda_{n_0+1})^{1/2},$$

where $m \leq \lambda \leq M < \tilde{\lambda}_1$, $(p = 1,2,\dots)$.

Theorem similar to theorems 4,5 also hold for second method (5). The methods (4),(5) seem to be very simple and convenient for computation. They can also be used for finding the extreme values m, $\tilde{\lambda}_1$ of the spectrum $\sigma(A)$.

R e f e r e n c e s :

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