Václav Havel
Construction of certain systems with two compositions

Commentationes Mathematicae Universitatis Carolinae, Vol. 6 (1965), No. 4, 413--428

Persistent URL: http://dml.cz/dmlcz/105033
A \textit{double quasigroup} is defined here as a triple \((S, +, \Box)\) where \(S, \ \text{card} \ S \geq 2\), is a set and \(+, \Box\) two binary compositions on \(S\) such that \((S, +)\) is a loop with a neutral element 0 satisfying \(x \Box 0 = 0 \Box x = 0\) for all \(x \in S\) and \((S \setminus \{0\}, \Box)\) is a quasigroup. If \((S \setminus \{0\}, \Box)\) has a neutral element, then \((S, +, \Box)\) is called \textit{double-loop} [4, p. 61].

Each double-quasigroup \((S, +, \Box)\) with a prescribed additive loop \((S, +)\) may be constructed as follows, [7a]: Let \((B, \circ)\) be the group of all bijective mappings of \(S\) onto \(S\), reproducing the element 0, with a natural composition \(\circ\); also, set \(0 : S \rightarrow \{0\}\). Choose any mapping \(\eta : S \rightarrow B \cup \{0\}\) satisfying \(\eta(0) = 0, \eta(x) = 0\) for \(x \in S \setminus \{0\}\), such that \(\eta(S \setminus \{0\})\) operates on \(S \setminus \{0\}\) simply transitively, and define the composition \(\Box\) on \(S\) by \(x \Box y = \eta(x) y\) for all \(x, y \in S\). Then \((S, +, \Box)\) is the double-quasigroup associated with \(\eta\), and every double-quasigroup \((S, +, \Box)\) with a prescribed additive loop may be obtained in this way.

Now we exhibit the familiar algebraical properties of a given \((S, +, \Box)\) in the following way:

\begin{align*}
A^+ \ (x + y) + z &= x + (y + z) \quad \text{for all} \ x, y, z \in S. \quad \text{(Associativity)} \\
C^+ \ x + y &= y + x \quad \text{for all} \ x, y \in S. \quad \text{(Commutativity)}
\end{align*}
\[ RD^+ \circ (y + x) = x \circ y + x \circ x \quad \text{for all } x, y, z \in S. \]  
(Right distributivity.)

\[ LD^+(x + y) = x \circ z + y \circ x \quad \text{for all } x, y, z \in S. \]  
(Left distributivity.)

\[ RP^+ \] Any equation \(-a \circ x + b \circ x = c\) has a unique solution \(x \in S\) for any given \(a, b, c \in S\) with \(a \neq b\).  
(Right planarity.)

\[ LP^+ \] Any equation \(x \circ a - x \circ b = c\) has a unique solution for any given \(a, b, c \in S\) with \(a \neq b\).  
(Left planarity.)

In the theory of incidence structures (partial planes) in the sense of [4, p.2] and in the theory of systems with generalized parallelity [7b] there are important the double-quasigroups satisfying the axioms \(A^+, \cdot LP^+ \circ A^+, RP^+\).

In the sequel we shall use modifications of the Moulton construction from the classical paper [1] (also see, e.g., [4], [5],[6]), and we wish to obtain some double-quasigroups \((S, +, \cdot)\) satisfying \(A^+, \cdot LP^+\circ A^+, RP^+\) respectively. It remains an open question whether double-quasigroups in which exactly one of the laws \(RP^+, \cdot LP^+\) holds are obtainable by this process.

We note that in a double-quasigroup \((S, +, \cdot)\) from \(A^+, \cdot RD^+\circ A^+, LD^+\) there follows \(LP^+\) or \(RP^+\) respectively.

A double-quasigroup \((S, +, \cdot)\) satisfying \(A^+, \cdot RP^+\) \(LP^+\) and either \(RD^+\circ A^+, LD^+\circ A^+\) is usually called a right or left quasifield, respectively, [4, p. 92].

We shall begin with the additive loop \((S, +)\) of a
double-quasigroup \((S,+,\cdot)\) and also a mapping \(\eta : S \to B\), and construct the associated double-quasigroup.

1. Let \(F = (S,+,\cdot)\) be a left quasifield and \(\Phi : S \to S\) a bijection with \(\Phi(0) = 0\). For arbitrary \(a \in S\) let \(\eta(a)\) be the mapping \(x \to \Phi(a)x, x \in S\); then the associated double-quasigroup \((S,+,\square)\) is also a left quasifield. \(RD^+\square\) holds if and only if \(\Phi\) is an additive mapping.

Proof. The validity of \(A^+, C^+\) in \((S,+,\square)\) is, of course, trivial. There is \(x \sqcap (y+z) = \Phi(x)(y+z) = \Phi(x)y + \Phi(x)z = x \sqcap y + x \sqcap z\), so that \(LD^+\square\) holds. Any equation \(-a \sqcap x + b \sqcap x = c\), for given \(a, b, c \in S \setminus \{0\}, a \neq b\), may be rewritten as \(-\Phi(a)x + \Phi(b)x = c\), and the unique solvability follows from \(RP^+\): If \(\Phi\) is not additive, there exist \(a_o, b_o \in S\) such that \(\Phi(a_o + b_o) \neq \Phi(a_o) + \Phi(b_o)\), and this implies \((a_o + b_o) \sqcap x = \Phi(a_o + b_o)x = \Phi(a_o)x + \Phi(b_o)x = a_o \sqcap x + b_o \sqcap x\) for all \(x \in S \setminus \{0\}\); thus \(RD^+\square\) is violated. If \(\Phi\) is additive, then \(RD^+\square\) follows directly.

One simple special case can be stated as follows: Let \(F\) be an ordered left quasifield \([4, p. 237]\); the set of all negative elements of \(F\) will be denoted by \(N\). We choose \(\Phi(a) = a\) for all \(a \geq 0\) and \(\Phi(N) = N\) (so that \(\Phi\) must map \(N\) bijectively onto \(N\)), and suppose \(\Phi(m) \neq m\) for some \(m \in N\). Then \(\Phi\) is not additive and the assumption of theorem 1 is fulfilled.
If \( \eta(a) \) is taken as \( x \rightarrow \theta(\phi(a), \psi(x)), x \in S \), where \( \phi, \psi, \theta \) are the bijections of \( S \) onto \( S \) with \( \phi(0) = \psi(0) = \theta(0) = 0 \), then the associated double-quasigroup \( (S, +, \square) \) fulfills \( LP^+\square \) and \( RP^+\square \) whereas \( LD^+\square \) or \( RD^+\square \) is satisfied precisely if \( \phi, \theta \) or \( \psi, \theta \) respectively are additive. This is the special case of the known notion of weak-isotopy, introduced in [3, p. 460]. In theorem 1 only a special case of this weak-isotopy was used. The connection between weak isotopic double-quasigroups and their associated systems with generalized parallelity [7b] can be investigated when the corresponding ternary composition \( T \) is introduced by
\[
T(a, b, c) = a \triangleleft b + c.
\]

If \( (S, +, \cdot) \) is a double-quasigroup, then we may choose \( \phi \) as the identity mapping on \( S \), \( \psi \) as the mapping \( x \rightarrow \alpha \triangleleft \beta, x \in S \), and \( \theta \) as the mapping \( x \rightarrow x / \beta, x \in S \); here \( \alpha, \beta \in S \setminus \{0\} \) are fixed elements, and \( \triangleleft \) and \( / \) denote, respectively the left and right division in \( (S \setminus \{0\}, \cdot) \). The associated double-quasigroup \( (S, +, \square) \) satisfies \( \alpha \square x = x \square \alpha = x \) for all \( x \in S \setminus \{0\} \), and was introduced in [3, p. 461] (but only under the assumption of \( LP^+\prime \) and \( RP^+\prime \) ) according to the description given by Hall.

For our aims, the most important special case of theorem 1 is that in which \( F = (S, +, \cdot) \) is a skew-field. If \( \phi \) is not additive, then \( (S, +, \square) \) is a proper left
quasifield without the identity element. According to [3, p. 463], in this manner the left quasifields, which are a form of "generalized natural field" [3, p. 451] of desarguesian planes, may be obtained.

2. Let $F = (S, +, \cdot)$ be a double-quasigroup. Take a mapping $\eta : S \to B \cup \{0\}$ with $\eta(0) = 0$ such that each $\eta(a), a \in S \setminus \{0\}$ has the form $x \mapsto a \Phi_a(x), \ x \in S$, where $\Phi_a : S \to S$ is an additive bijection with $\Phi_a(0) = 0$ and $\eta(S \setminus \{0\})$ acts simply transitively on $S \setminus \{0\}$. Then the associated double-quasigroup $(S, +, \cdot)$ satisfies $RD^+ \circ$; the axiom $RP^+ \circ$ is fulfilled precisely if the mappings $x \mapsto -a \Phi_a(x) + b \Phi_b(x), \ x \in S$ are bijective for all distinct $a, b$ of $S \setminus \{0\}$.

Proof. We have $x \circ (y + z) = x \Phi_a(y) + x \Phi_a(z) = x \Phi_a(y) + x \Phi_a(z)$ for all $x, y, z \in S$, so that $RD^+ \circ$ holds. The rest of the theorem is obvious.

The André quasifield [4, p. 206] is constructed as described in theorem 2 on taking a field for $F$, and $\Phi_a, a \in S \setminus \{0\}$, as suitable automorphisms of $F$ leaving fixed each element of some proper subfield of $F$.

3a. Let $F = (S, +, \cdot)$ be an ordered non-commutative field and $\Phi$ and additive bijective mapping of $S$ onto $S$ satisfying $x > 0 \Rightarrow \Phi(x) > 0$ and $x < 0 \Rightarrow \Phi(x) < 0$. Define $\eta(a)$ as the mapping $x \mapsto a \cdot x, x \in S$ if $a \geq 0$, and as the mapping $x \mapsto \Phi(x) \cdot a, x \in S$ if $a < 0$. 

- 417 -
Then the associated double-quasigroup \(( S, +, \circ \) satisfies \( A^+, C^+, RD^+ \circ \) (and thus also \( LP^+ \circ \) ) and does not satisfy \( LD^+ \circ \). Moreover, \( RP^+ \circ \) holds if and only if the mappings \( x \mapsto -a \cdot x + \tilde{\phi}(x) \cdot b \), \( x \in S \), for \( a > 0 > b \) and \( x \mapsto -\tilde{\phi}(x) \cdot a + b \cdot x \), \( x \in S \) for \( a < 0 < b \) are bijections of \( S \) onto \( S \).

**Proof.** For \( x \geq 0 \) we have \( x \circ (y+z) = x \circ y + x \circ z \), and for \( x < 0 \) we have \( x \circ (y+z) = \tilde{\phi}(y+z) \cdot x = (\tilde{\phi}(y) + +\tilde{\phi}(z)) \cdot x = x \circ y + x \circ z \), so that \( RD^+ \circ \) is valid. If we choose \( x_o, y_o, z_o \in S \) such that \( x_o > 0 > y_o \), \( x_o + y_o > 0 \), \( z_o = 1 \), then \((x_o+y_o) \circ z_o = x_o \cdot z_o + y_o \cdot z_o + +x_o \cdot z_o + \tilde{\phi}(z_o) \cdot y_o = x_o \circ z_o + y_o \circ z_o ; \) thus \( LD^+ \circ \) is violated. It remains to examine a mapping \( x \mapsto -a \circ x + b \circ x \), \( x \in S \) for given distinct elements \( a, b \) of \( S \setminus \{0\} \).

We distinguish the following four alternatives:
- \( -a_o \circ x + b \circ x = -a \cdot x + \tilde{\phi}(x) \cdot b \) for \( a > 0 \), \( b < 0 \);
- \( -a_o \circ x + b \circ x = (-a + b) \cdot x \) for \( a > 0 \), \( b > 0 \);
- \( -a_o \circ x + b \circ x = \tilde{\phi}(x) \cdot (-a + b) \) for \( a < 0 \), \( b < 0 \);
- \( -a_o \circ x + b \circ x = -\tilde{\phi}(x) \cdot a + b \cdot x \) for \( a < 0 \), \( b > 0 \).

In the second and third cases the required bijectivity is easily obtained; the first and fourth alternative figure explicitly in the last condition of the theorem.

3b. Let \( F = (S, +, \cdot) \) be an ordered non-commutative \( S \) -field, and \( \tilde{\phi} \) an order preserving mapping of \( S \) onto \( S \) with \( \tilde{\phi}(0) = 0 \). Define \( \eta(a) \) as the mapping \( x \mapsto a \cdot x \), \( x \in S \), if \( a \geq 0 \), and \( x \mapsto \tilde{\phi}(x) \cdot a \), \( x \in S \), if \( a < 0 \).
Then the associated double-quasigroup \((S, +, \square)\) satisfies \(A^+, C^+, L P^+\) and does not satisfy \(R D^+\). Moreover \(R D^+\) holds if and only if the mapping \(x \rightarrow -a \cdot x + \psi(x) \cdot b, x \in S\), for \(a > 0 > b\) and \(x \rightarrow -\Phi(x) \cdot a + b \cdot x, x \in S\) for \(a < 0 < b\) are surjections of \(S\) onto \(S\).

The proof is analogous to that of theorem 3a with the exception of the axiom \(L P^+\). But any mapping \(x \rightarrow x \cdot a - x \cdot b, x \in S\) has the form \(x \rightarrow x \cdot a - x \cdot b = x \cdot (a - b)\) for \(x \geq 0\) and \(x \rightarrow \Phi(a) \cdot x \rightarrow \Phi(b) \cdot x = (\Phi(a) - \Phi(b)) \cdot x\) for \(x < 0\).

From the order-preservation of \(\Phi\) there follows bijectivity of the mapping considered. At the end of the theorem, we have utilized surjectivity, this being possible because \(\Phi\) is order-preserving.

For the construction of a non-bijective mapping \(x \rightarrow a \cdot x + \psi(x) \cdot \alpha, x \in S\), for some positive \(\alpha\) (if such situation occurs at all), the known ordered non-commutative field of Hilbert does not seem to be sufficiently general.

4. Let \(F = (S, +, \cdot)\) be an ordered field and let \(N\) be the set of all negative elements of \(F\). We denote by \(\Phi : N \rightarrow N\) an order-preserving bijection and \(\psi : S \rightarrow S\) an order-preserving bijection with \(\psi(0) = 0\). Let \(\eta(a)\) be the mapping \(x \rightarrow \psi(a) \cdot x, x \in S\), if \(a \geq 0\) and \(x \rightarrow \psi(a) \cdot x, x \geq 0\) and \(x \rightarrow \Phi(a) \cdot x, x < 0\) if \(a < 0\). The associated double-quasigroup \((S, +, \square)\) satisfies \(A^+, C^+, R P^+\). Moreover, \(L P^+\) holds if and only if the mappings \(x \rightarrow \psi(x) \cdot a - \Phi(x) \cdot b, x \in N\) for \(a > 0, b < 0\), and \(x \rightarrow \Phi(x) \cdot a - \psi(x) \cdot b, x \in N\) for \(a < 0, b > 0\), and
$a < 0, b > 0$ are surjections of $N$ onto $N$.

Proof. Consider the mapping $x \rightarrow -a \circ x + b \circ x, x \in S$, for given distinct $a, b$ of $S \setminus \{0\}$. Without loss of generality we may restrict ourselves to the case $a < 0, a < b$.

Then we distinguish three cases

$-a \circ x + b \circ x = (-\Phi(a) + \Psi(b)) \cdot x$ for $x < 0, b \geq 0$, 
$-a \circ x + b \circ x = (-\Phi(a) + \Phi(b)) \cdot x$ for $x < 0, b < 0$, 
$-a \circ x + b \circ x = (-\Phi(a) + \Psi(b)) \cdot x$ for $x \geq 0$.

Since $\Phi$ and $\Psi$ are order-preserving and $\Psi(0) = 0$; the considered mapping is bijective. Analogously, consider the mapping $x \rightarrow x \circ a - x \circ b, x \in S$, where one may suppose without loss of generality, that $b < 0, a > b$. Then we distinguish three alternatives:

$\begin{align*}
x \circ a - x \circ b &= \Psi(x) \cdot a - \Phi(x) \cdot b &\text{for} & x < 0, a > 0; \\
x \circ a - x \circ b &= \Phi(x) \cdot (a - b) &\text{for} & x < 0, a < 0; \\
x \circ a - x \circ b &= \Psi(x) \cdot (a - b) &\text{for} & x \geq 0.
\end{align*}$

In the second and the third case the required bijectivity follows directly, and in the first case it is stated in the last condition of the theorem. As $\Phi$ and $\Psi$ are order-preserving, bijectivity can be replaced by surjectivity. From this the rest of the proof follows. The bijection $\Phi$ and $\Psi$ can be chosen in such a way that $RD^+ \circ$ and $LD^+ \circ$ are both violated [6, pp. 93-94].

If $\psi(x)$ for $x \in S$ and $\phi(x) = \nu \cdot x$ for $x \in N$ for fixed $\nu > 0$, we obtain the classical case of the construction, especially the initial case of [1].

5. Let $F = (S, +, \cdot)$ be a pseudoordered field [6, p. 427], and denote by $N$ the set of all negative elements of $F$.
Let $\phi : S \to S$, $\psi : S \to S$ be pseudoorder-preserving bijections \cite{6, p. 428} with $\phi(0) = 0$ and $\psi(0) = 0$.

Suppose that $\eta(a)$ is a mapping $x \mapsto \psi(a) \cdot x$, $x \geq 0$

or $x \mapsto \phi(a) \cdot x$, $x \leq 0$ for every $a \in S$.

Then the associated double-quasigroup $(S, +, \cdot)$ satisfies $A^+$, $C^+$, $R P^+$.

Moreover, $L P^+$ holds if and only if any mapping $x \mapsto \phi(x) \cdot a - \psi(x) \cdot b$, $x \in S$ for given $a, b$ with opposite signs (in the sense of \cite{6, p. 427}) is a surjection of $S$ onto $S$.

Proof. The validity of $R P^+$ must be obtained in a manner different from that of the proof of theorem 4.

Following \cite{6, p. 90}, we replace the requirement of the unique solvability in $R P^+$ by requiring only the existence of solutions

\[ a \circ c - a \circ d = b \circ c - b \circ d \Rightarrow c = d \text{ for } a, b, c, d \in S; \]

if $c \geq 0$, $d \geq 0$, then $a \circ c - a \circ d = b \circ c - b \circ d \Rightarrow \psi(a) \cdot (c - d) = \psi(b) \cdot (c - d)$;

if $c < 0$, $d < 0$, then $a \circ c - a \circ d = b \circ c - b \circ d \Rightarrow \phi(a) \cdot (c - d) = \phi(b) \cdot (c - d)$;

if $c < 0$, $d \geq 0$, then $a \circ c - a \circ d = b \circ c - b \circ d \Rightarrow (\phi(a) - \phi(b)) \cdot c = (\psi(a) - \psi(b)) \cdot d$;

and if $c \geq 0$, $d < 0$, then $a \circ c - a \circ d = b \circ c - b \circ d \Rightarrow (\psi(a) - \psi(b)) \cdot c = (\phi(a) - \phi(b)) \cdot d$.

In the first and the second case $c = d$ follows,

whereas in the third case \( \frac{c}{d} < 0 \Rightarrow \frac{\phi(a) - \phi(b)}{\psi(a) - \psi(b)} = \frac{\phi(a) - \phi(b)}{a - b} \cdot \frac{\psi(a) - \psi(b)}{a - b} \).
and one of the mappings $\phi$, $\psi$ cannot be pseudoorder-preserving, contradicting the hypothesis. The fourth case may be studied analogously. Thus the condition $U$ holds in $(S, +, \square)$. We verify that any equation $a \square x - b \square x = c$ has at least one solution $x \in S$ for given $a, b, c \in S$, $a + b$. Indeed, for $x \geq 0$ this equation can be rewritten as $(\psi(a) - \psi(b)) \cdot x = c$, thus if $\frac{c}{\psi(a) - \psi(b)} > 0$, we may use the solution $x = \frac{c}{\psi(a) - \psi(b)}$. For $x < 0$ one may rewrite as $(\phi(a) - \phi(b)) \cdot x = c$, so that for $\frac{c}{\phi(a) - \phi(b)} < 0$ we may put $x = \frac{c}{\phi(a) - \phi(b)}$. It is clear that $\frac{c}{\psi(a) - \psi(b)} = \frac{c}{a - b} : \frac{\psi(a) - \psi(b)}{a - b} > 0 \iff \frac{c}{\phi(a) - \phi(b)} > 0$,
while in the contrary case one of the mappings $\phi$, $\psi$ is not pseudoorder-preserving.

Finally, we investigate any equation $x \square a - x \square b = c$ for given $a, b, c \in S$, $a + b$. For $a \geq 0, b \geq 0$ or for $a < 0, b < 0$ we have $\psi(x) \cdot (a - b) = c$ or $\phi(x) \cdot (a - b) = c$ respectively, and the unique solvability follows from the definition of $\psi$ and $\phi$. The remaining cases $a \geq 0, b < 0$ and $a < 0, b \geq 0$ yield the equations $\psi(x) \cdot a - \phi(x) \cdot b = c$ and $\phi(x) \cdot a - \phi(x) \cdot b = c$ respectively, stated in the last condition of our theorem.

If we neglect the postulate of unique solvability for $x \in S \setminus \{0\}$ or for $y \in S \setminus \{0\}$ of the equation $x \square y = x$ for given $y, z \in S \setminus \{0\}$ or $x, z \in S \setminus \{0\}$ respectively, then we may construct, by the method -
of theorem 5, a system $(S, +, \square)$ such that $(S, +)$ is an Abelian group with neutral element $0$, $\forall x \in S \land (0 \circ x) = 0$ for all $x \in S$ and $(S \setminus \{0\}, \square)$ is a groupoid satisfying condition $U$. In the assumptions of theorem 5 it is sufficient to replace the requirement that $\phi$ be a pseudo-order-bijection by that $\phi$ is to be a pseudoorder-injection. Then the resulting $(S, +, \square)$ satisfies $A^r, C^r, U, R^p$ and does not satisfy $L^p$. To obtain a concrete case choose $F = (S, +, \cdot)$ to be the field $F_0 = \left(\mathbb{Q}(\xi)\right)$ of rational expressions over the basic field $F_0 = (S_0, +, \cdot)$ and define the pseudoorder on $F$ as follows \cite[p. 428]{6}: if
\[
a = \frac{f(\xi)}{g(\xi)} \in S \quad \text{has the lowest form with non-zero polynomials} \ f(\xi), g(\xi), \quad \text{then set} \ x > 0 \ \text{or} \ x < 0 \ \text{according as} \ \deg f(\xi) - \deg g(\xi) \ \text{is even or odd.}
\]
Next, choose $\tilde{\phi}(a) = a^3$, $a \in S$, and $\tilde{\psi}(a), a \in S$; it may be shown that $\tilde{\phi}$ is pseudoorder-preserving injection which is not a surjection and the same conclusion holds for the mapping $x \to 1 \cdot \psi(x) + 1 \cdot \tilde{\phi}(x) = x + x^3$, $x \in S$ (e.g. for $\xi$ there is no $x$ such that $x^3 = \xi$ or $x + x^3 = \xi$). Another example is obtained if $F = (S, +, \cdot)$ is the rational field with the following pseudoorder \cite[p. 427]{6}: choose some prime $\mu$ and express every rational in the form $\mu^a \frac{a}{b}$ where $a, b$ are the lowest integers prime to $\mu$, and then say that this rational is positive or negative according as $\mu$ is even or odd. Now set $\psi(a) = a$, $a \in S$ and $\tilde{\phi}(a) = a^3$, $a \in S$. It may be proved that $\tilde{\phi}$ is a pseudoorder-preserving injection which is not a
surjection and that also the mapping \( x \rightarrow 1 \cdot \psi(x) + 1 \cdot \phi(x) = x + x^3 \) is of the same type. The so-obtained systems 
\((S, +, \Box)\) may be interpreted as near-planar ternary rings, which are not planar (see the following definition) if the 
corresponding ternary composition \( T \) on \( S \) is introduced 
by \( T(x, \mu, \nu) = x \Box \mu + \nu \) for all \( x, \mu, \nu \in S \).

Now we use theorem 5 for rational field \( F = (S, +, \cdot) \) 
with the pseudoorder described above and put \( \psi (\alpha) = \alpha \), 
\( \alpha \in S \), and \( \phi (\mu^m \frac{a_1}{a_2}) = \mu^m \frac{a_1}{a_2} \) for \( \mu^m \frac{a_1}{a_2} \) 
in canonical form in \( S \setminus \{ 0 \} \), whereas \( \phi (0) = 0 \).

Then \( \phi \) is a pseudoorder-preserving bijection because for \( a = \mu^m \frac{a_1}{a_2} \), \( \beta = \mu^m \frac{b_1}{b_2} \in S \setminus \{ 0 \} \) with \( m - n \geq 0 \) 
there is \( \frac{\phi(a) - \phi(\beta)}{a - \beta} = \mu^m \frac{a_1 b_2 - a_2 b_1}{a_2 b_2} = \mu^m \frac{a_1 b_2 - a_2 b_1}{a_2 b_2} > 0 \).

Then, for \( \mu = 2 \), the mapping \( x \rightarrow x + \phi(x), x \in S \), 
is not surjective since the equation 
\[ 2^m \frac{x_4}{x_2} + 2^m \frac{x_5}{x_1} = \]
\[ = 2^m (-1) \Leftrightarrow \left( \frac{x_4}{x_2} \right)^2 + 2^{1-m} \left( \frac{x_5}{x_2} \right) + 1 = 0 \] has only a non-rational 
solution \( \frac{x_5}{x_2} = 2^m \pm \sqrt{2^{-2m-1}}, m = 0, \pm 1, \pm 2, \ldots \); the ele-
ment \( 2^m (-1) \in S \) does not have an inverse image with 
regard to \( \phi \). The obtained system \((S, +, \Box)\) can be in-
terpreted as a near-planar ternary ring which is not planar.

\[ \text{m) The existence of such } \phi \text{ was orally communicated to me by} \]
\[ \text{O. Kowalski.} \]
(see the following definition) if the corresponding ternary composition \( T \) on \( S \) is introduced by
\[
T(x, u, v) = x \circ u + v
\]
for all \( x, u, v \in S \).

This ternary ring satisfies the condition of "symmetry":
\[
T(x, u, v) = y \text{ is uniquely solvable in } x \in S \text{ for given } u, v, y \in S, \ u \neq 0.
\]
The existence of such ternary rings is important because it shows that the notion of symmetric near-planar ternary rings ([7b]) is in fact more general than that of planar ternary rings.

By a ternary ring \( (S, T) \) is meant here a non-empty set \( S \) with a ternary composition on \( T \) satisfying
\[
T(S, S, S) = S.
\]
The ternary ring \( (S, T) \) is called near-planar if
1° there exists an element \( 0 \in S \) such that
\[
T(x, 0, v) = v, \ T(0, u, v) = v \text{ for all } x, u, v \in S,
\]
2° any equation \( T(a, b, v) = d \) is, for given \( a, b, d \in S \), uniquely solvable in \( v \in S \),
3° for given \( x_1, y_1, x_2, y_2 \in S \) with \( x_1 \neq x_2 \), the equations \( T(x_i, u, v) = y_i, i = 1, 2 \) have a unique solution \( x \in S \).

The near-planar ternary ring \( (S, T) \) is said to be planar if
\( 4^0 \) for given \( u_1, v_1, u_2, v_2 \in S \) with \( u_1 \neq u_2 \) the equation \( T(x, u_1, v_1) = T(x, u_2, v_2) \) has a unique solution \( x \in S \).

6. Let \( F = (S, +, \cdot) \) be a pseudoordered field and \( \Phi : S \rightarrow S \) a bijection with fixed element \( 0 \). We define a ternary composition \( T \) on \( S \) as follows:
The ternary ring satisfying 1° and 2°; moreover 3° holds precisely if $\Phi$ is a pseudoorder-monotone (in the sense of [6, p. 428]).

Proof. According to the definition of $\Phi$ and $T$, 0 satisfies condition 1°. Condition 2° is obvious for $\mu \geq 0$ and follows from the bijectivity of $\Phi$ if $\mu < 0$. Given the equations $y_i = T(x_i, \mu, \nu)$, $i = 1, 2$, with $x_1, y_1, x_2, y_2 \in S$, $x_1 \neq x_2$, $y_1 \neq y_2$, we distinguish two cases:

1. $y_i = x_i \cdot \mu + \nu$, $i = 1, 2$ for $\mu \geq 0$,
2. $\Phi(y_i) = \Phi(x_i) \cdot \mu + \nu$, $i = 1, 2$ for $\mu < 0$.

Thus from (1) there follows $(x_1 - x_2) \cdot \mu = y_1 - y_2$, $\sigma_g(y_1 - y_2) = \sigma_g(y_1 - y_2)$ and from (2) there follows $(\Phi(x_1) - \Phi(x_2)) \cdot \mu = \Phi(y_1) - \Phi(y_2), \sigma_g(\Phi(x_1) - \Phi(x_2)) = \sigma_g(\Phi(y_1) - \Phi(y_2))$. We conclude that 3° is satisfied precisely if

$$\frac{y_1 - y_2}{x_1 - x_2} > 0 \iff \frac{\Phi(x_1) - \Phi(x_2)}{\Phi(y_1) - \Phi(y_2)} > 0 \text{ or } \frac{\Phi(x_1) - \Phi(x_2)}{\Phi(y_1) - \Phi(y_2)} > 0 \text{ or } \frac{\Phi(x_1) - \Phi(x_2)}{x_1 - x_2} = \sigma_g \frac{\Phi(y_1) - \Phi(y_2)}{y_1 - y_2}.$$

all of which mean that $\Phi$ is a pseudoorder-monotone. Condition 4° holds in $(S, T)$ precisely if for $\mu_1 < \mu < \mu_2$ each $\Phi(x) \cdot \mu_1 + \nu = \Phi(x \cdot \mu_2 + \nu_2)$ is uniquely solvable in $x \in S$. For $F$ the real field and $\Phi(x) = x^3$, $x \in S$, we obtain the situation investigated in [2].

7. Let $F = (S, +, \cdot)$ be a pseudoorder field and $\Phi : S \rightarrow S$ a bijection with $\Phi(0) = 0$; let $T$ be
the ternary composition on \( S \) defined as follows:
\[
T(x, u, v) = \phi(x) - u + v \quad \text{for} \quad u \geq 0 \quad \text{and} \quad T(x, u, v) = \phi^{-1}(x \cdot (u + \phi(v))) \quad \text{for} \quad u < 0.
\]
Then \((S, T)\) is a ternary ring satisfying 1° and 2°; moreover 3° holds precisely if \( \phi \) is pseudoorder-monotone.

Proof. Condition 1° is obviously satisfied. Condition 2° is valid for \( u \geq 0 \) trivially, and for \( u < 0 \) follows from bijectivity of \( \phi \). Thus we need only consider condition 3°: assume given \( x_1, y_1, x_2, y_2 \in S \), \( x_1 \neq x_2, y_1 \neq y_2 \), and distinguish two alternatives:

(3) \( y_i = \phi(x_i) \cdot u + v, \quad i = 1, 2 \quad \text{for} \quad u \geq 0 \).
(4) \( \phi(y_i) = x_i \cdot u + \phi(v), \quad i = 1, 2 \quad \text{for} \quad u < 0 \).

From (3) there follows \( \phi(x_1) - \phi(x_2) \cdot u = y_1 - y_2 \), so that \( \frac{\phi(x_1) - \phi(x_2)}{y_1 - y_2} > 0 \); from (4) there follows

\[
(x_1 - x_2) \cdot u = \phi(y_1) - \phi(y_2), \quad \text{so that} \quad \frac{\phi(y_1) - \phi(y_2)}{x_1 - x_2} < 0.
\]

We conclude that \( \frac{\phi(x_1) - \phi(x_2)}{y_1 - y_2} \) and \( \frac{\phi(y_1) - \phi(y_2)}{x_1 - x_2} \) simultaneously have the same sign, which implies that \( \phi \) is pseudoorder-monotone (and conversely). Condition 4° holds in \((S, T)\) if and only if, for \( u_1 < 0 < u_2 \), each \( x \cdot u_1 + \phi(v_1) = \phi(\phi(x) \cdot u_2 + v_2) \) is uniquely solvable in \( x \in S \). If \( F \) is taken to be \( \mathbb{Q} \) or \( \mathbb{R} \) chosen according to André’s procedure [5, p. 204–205], one obtains the planar ternary ring investigated in [5].

Literature:


(Received August 23, 1965)