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MODEL $\nabla[\omega_\alpha \rightarrow \omega_\beta]$ IN WHICH β IS LIMIT NUMBER

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In [3], some models $\nabla[\omega_\alpha \rightarrow \omega_{\beta+1}]$ were constructed. In this note the discussion of existence of models $\nabla[\omega_\alpha \rightarrow \omega_\beta]$ is performed for the case that β is a limit number.

We shall use the notation from [1],[2],[3].

(1) Metadefinition: A model ∇ of the Gödel-Bernays set theory Σ is said to be of type $[\omega_\alpha \rightarrow \omega_\beta]$ if the following holds in the set theory: $\omega_\alpha \in \omega_\beta$, and

(a) if f is a relatively cardinal number of the model ∇ , $f \in^* \aleph_{\omega_\alpha}$ or $\aleph_{\omega_\beta} \in^* f$ then f is a cardinal number of the model ∇ ,

(b) \aleph_{ω_β} is the first cardinal number of the model ∇ greater than \aleph_{ω_α} . (See [3].)

(2) Metatheorem: If the singularity of ω_β is provable, then there exists no model of type $[\omega_\alpha \rightarrow \omega_\beta]$.

Proof. In such a model $\aleph_{\omega_\alpha} \in^* f \in^* \aleph_{\omega_\beta}$ implies $\overline{f}^* \in^* \aleph_{\omega_\alpha}$. But ω_β is a singular number, hence there is an a such that $a \subseteq \omega_\beta$, $\overline{a} \in \omega_\beta$, $\omega_\beta = \bigcup a$. From this we obtain easily $\aleph_{\omega_\beta} \in^* \bigcup^* \aleph_a$, $\overline{\aleph_a}^* \in^* \aleph_{\omega_\alpha}$, $(f)[f \in^* \aleph_a \rightarrow \overline{f}^* \in^* \aleph_{\omega_\alpha}]$. Hence $\overline{\aleph_{\omega_\beta}}^* \in^* \aleph_{\omega_\alpha}$, a contradiction.

In the following, we assume the generalized continuum hypothesis to be valid in the set theory. Let \mathcal{A} be an in-

accessible cardinal number of the set theory, θ a set of power \mathfrak{v} . Let \mathcal{I} be an ideal on θ such that there is a $\omega_\gamma \in \mathfrak{v}$ with $(x)[x \in \mathcal{I} \rightarrow \bar{x} \in \omega_\gamma]$. For every $x \in \theta$ let $\langle c(x), t(x) \rangle$ be a topological space, and $\langle \prod_{x \in \theta} c(x), \prod_{x \in \theta} t(x) \rangle$ the topological product of the spaces $\langle c(x), t(x) \rangle$ by the ideal \mathcal{I} as defined in [3]. If $\langle c, t \rangle$ is a topological space, we define $\mu(c, t) = \min\{\omega_\gamma; \neg(\exists a)[\exists x(a) \& a = t \& \bar{a} = \omega_\gamma]\}$ (see [2]), $\chi_t(c, t) = \min\{\omega_\gamma; \omega_\gamma = \bar{b}, b \text{ some basis of } \langle c, t \rangle\}$.

(3) Theorem. Let $\chi_t(\langle c(x), t(x) \rangle) \in \mathfrak{v}$ for every $x \in \theta$. Then $\mu(\langle \prod_{x \in \theta} c(x), \prod_{x \in \theta} t(x) \rangle) \in \mathfrak{v}$.

Proof. Let \mathcal{a} be a system of open sets in the space $\langle \prod_{x \in \theta} c(x), \prod_{x \in \theta} t(x) \rangle$, $\exists x(a), \bar{a} = \mathfrak{v}$. Denote by $\beta(x)$ some basis composed of open sets in the space $\langle c(x), t(x) \rangle$, such that $\overline{\beta(x)} = \chi_t(\langle c(x), t(x) \rangle)$. In $\langle \prod_{x \in \theta} c(x), \prod_{x \in \theta} t(x) \rangle$ choose the basis generated by the sets \bar{q} with $q \in \mathcal{L}^{\mathcal{I}} \beta(x)$ (see [2], def. 3,4). We may assume that \mathcal{a} contains elements of this basis only. Define $(*) f \in \mathcal{L} \equiv \exists n c f \& \mathcal{D}(f) \subset \theta \& \overline{\mathcal{D}(f)} \in \omega_\gamma \& (x)[x \in \mathcal{D}(f) \rightarrow \rightarrow f(x) \in \beta(x)] \& (\exists g)[\bar{g} \in \mathcal{a} \& (x)(x \in \mathcal{D}(f) \cap \mathcal{D}(g) \rightarrow \rightarrow f(x) = g(x))]$.

The axiom of choice implies the existence of a function B such that

$$\exists n c B \& \mathcal{D}(B) = \mathcal{L} \& (f)\{f \in \mathcal{D}(B) \rightarrow [\overline{B(f)} \in \mathcal{a} \& \& (x)(x \in \mathcal{D}(f) \cap \mathcal{D}(B(f)) \rightarrow f(x) = (B(f))(x))]\}$$

Now choose $\bar{g}_0 \in \mathcal{a}$ and put $G_0 = \{g_0\}$. Having

defined G_ρ for all $\rho \in \sigma \in \omega_{\gamma+1}$, we define G_σ by $(**)$ $g \in G_\sigma \equiv (\exists f)[f \in \mathcal{b} \& \mathcal{D}(f) \subseteq \bigcup_{\rho \in \sigma} G_\rho \& g = B(f)]$, and $G = \bigcup_{\sigma \in \omega_{\gamma+1}} G_\sigma$. By induction one proves easily that $\overline{G}_\sigma \in \mathcal{V}$, $\overline{G} \in \mathcal{V}$. Then there must exist a $\overline{h} \in \mathcal{a}$ with $h \notin G$. Set $y = \mathcal{D}(h) \cap \mathcal{D} \cup G$, then $y \neq \emptyset$ because $\overline{h} \cap \overline{G}_\sigma = \emptyset$, also $\overline{y} \in \omega_\gamma$. Set $h_0 = h \wedge y$. From $(*)$ we obtain $h_0 \in \mathcal{b}$, and there is a σ such that $\mathcal{D}(h_0) = y = \bigcup_{\rho \in \sigma} G_\rho$, then $B(h_0) \in G_\sigma \subset G$ is an obvious consequence of $(**)$. Next, $h \neq B(h_0)$ implies $\overline{h} \cap \overline{B(h_0)} \neq \emptyset$, because $(x)[x \in \mathcal{D}(h) \cap \mathcal{D}(B(h_0)) \rightarrow h(x) = (B(h_0))(x)]$. Hence $h = B(h_0) \in G$. This is a contradiction.

We shall construct models $\nabla[\omega_\alpha \rightarrow \mathcal{V}]$ under the assumption that ω_α is a regular cardinal. The parameters of the model ∇ will be chosen as follows:

Set $\Theta = \{L; \omega_\alpha \in L \in \mathcal{V}\}$, and define:

I. $ind = \{a_L; L \in \Theta\}$,

II. $G(a_L) = \omega_L \times \omega_\alpha$ for every $L \in \Theta$.

III. Definition of the space $\langle c, t \rangle$: for every

$L \in \Theta$ define

$f \in c(L) \equiv \text{Fnc } f \& \mathcal{D}(f) = \omega_\alpha \& \mathcal{W}(f) \subseteq \omega_L$

$f \in \mathcal{b}(L) \equiv \text{Fnc } f \& \mathcal{D}(f) \subseteq \omega_\alpha \& \mathcal{W}(f) \subseteq \omega_L \& \overline{\mathcal{D}(f)} \in \omega_\alpha$

$\overline{f} = \{g; g \in c(L) \& f \subseteq g\}$ for $f \in \mathcal{b}(L)$,

$t(L)$ is the topology on $c(L)$ generated by basis $\mathcal{b}(L)$.

Let \mathcal{l} be the ideal: $x \in \mathcal{l} \equiv x \subseteq \Theta \& \overline{x} \in \omega_\alpha$ and $\langle c, t \rangle$ the space $\langle \bigcup_{L \in \Theta} c(L), \bigcup_{L \in \Theta} t(L) \rangle$.

IV. For $L \in \Theta$ and $\langle \gamma \sigma \rangle \in \omega_L \times \omega_\alpha$ we define

$$\kappa(\langle \gamma \sigma \rangle a_\iota) = \{f; f \in c \ \& \ (f(\iota))(\sigma) = \gamma\}.$$

It is then easy to prove the following statements:

For every $\iota \in \theta$ there is $\gamma_\iota(\langle c(\iota), t(\iota) \rangle) \in \mathcal{V}$, and then by theorem (3),

$$(1) \ \mu(c, t) \in \mathcal{V},$$

(2) $\omega_\alpha \in \nu(c, t)$ (since the intersection of a monotone system of ω_β sets from the basis of the space $\langle c, t \rangle$ is open set, if $\omega_\beta \in \omega_\alpha$, and it is non-void set, if $\omega_\beta = \omega_\alpha$).

By [2] theorem 4, condition (a) from metadefinition (1) holds in the model $\nabla(\text{ind}, G, \langle c, t \rangle, \kappa, \mathcal{J})$.

Obviously h_{a_ι} is a 1-1 mapping of h_{ω_α} onto h_{ω_ι} in the model ∇ .

Hence condition (b) from metadefinition (1) also holds in the model ∇ .

L i t e r a t u r e :

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