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ON THE SOLUTION OF THE MIXED PROBLEM

Jan KADLEC and Alois KUFNER, Praha

(Preliminary communication)

1.

Let Ω be a bounded domain in the plane E_2, whose boundary $\partial\Omega$ fulfils locally a Lipschitz condition. Decompose the boundary $\partial\Omega$ into two parts,

$$\partial\Omega = \Gamma_1 + \Gamma_2 ,$$

where Γ_1 has positive measure. Consider a function φ on $\partial\Omega$ such that

$$\varphi = 0 \quad \text{on} \quad \Gamma_1 ,$$

$$\varphi > 0 \quad \text{on} \quad \Gamma_2 .$$

Let

(1.1) $$Au = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x_1, x_2)\frac{\partial u}{\partial x_j}\right) + c(x_1, x_2)u$$

be an elliptic differential operator of the second order, $n = (n_1, n_2)$ the exterior normal vector to $\partial\Omega$ and

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i} n_j$$

the exterior co-normal derivative.

In this preliminary communication, we shall state some results concerning the solution of the mixed problem

(1.2) $$Au = f \quad \text{in} \quad \Omega ,$$

(1.3) $$u + \varphi\frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega .$$

It will be pointed out that, under further assumptions, the solution may be sought in special weight spaces, with the weight function

$$[dist\,(x\,,\,\Gamma_1)]^{\alpha}\ ;$$

these make it possible to give a better characterization of the behavior of solutions in the neighborhood of those points on $\partial\Omega$ which are limit points of both Γ_1 and Γ_2 .

From this point of view it is possible to solve the mixed problem also for those right-hand sides and boundary conditions for which the variational solution cannot be found without using weight functions (i.e. there exists no solution in the corresponding space with $\alpha = 0$). Furthermore, one can (for various f and g) find better solutions than by the usual variational procedure.

<u>Remark</u>: The fact that only the two-dimensional case is considered, is not essential; in n dimensions the difficulties are only in describing the position and shape of the parts Γ_1 and Γ_2 of the boundary $\partial\Omega$.

2.

In this section we shall introduce some functional spaces. For simplicity we consider only real functions and functionals; derivatives are understood in the sense of distribution-theory.

The space of all functions u for which the norm

$$(2.1)\quad \|u\|_{W_2^{(1)}(\Omega)} = \left(\|u\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x_1}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_2(\Omega)}\right)^{1/2}$$

is finite will be denoted by $W_2^{(1)}(\Omega)$.

Let $\rho(x)$ be the distance between the point $x = (x_1, x_2)$ and Γ_1, and let α be a real number. It will be said that the function u is in the space $L_{2,\alpha}(\Omega)$ if

$$\|u\|_{L_{2,\alpha}(\Omega)} = \|u\rho^{\alpha/2}\|_{L_2(\Omega)} = \left(\int_\Omega |u(x)|^2 \rho^\alpha(x)\, dx\right)^{1/2} .$$

Denote by $W_{2,\alpha}^{(1)}(\Omega)$ the set of all functions with the finite norm

$$(2.2)\quad \|u\|_{W_{2,\alpha}^{(1)}(\Omega)} = \left(\|u\|^2_{L_{2,\alpha}(\Omega)} + \left\|\frac{\partial u}{\partial x_1}\right\|^2_{L_{2,\alpha}(\Omega)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_{2,\alpha}(\Omega)}\right)^{1/2} .$$

Next, let $V_{2,\alpha}^{(1)}(\Omega)$ be the space of all functions such that

$$u \in L_{2,\alpha-2}(\Omega),\ \frac{\partial u}{\partial x_i} \in L_{2,\alpha}(\Omega) \qquad (i = 1, 2) ,$$

with the corresponding norm

$$(2.3)\quad \|u\|_{V_{2,\alpha}^{(1)}(\Omega)} = \left(\|u\|^2_{L_{2,\alpha-2}(\Omega)} + \left\|\frac{\partial u}{\partial x_1}\right\|^2_{L_{2,\alpha}(\Omega)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_{2,\alpha}(\Omega)}\right)^{1/2} .$$

Obviously $V_{2,\alpha}^{(1)}(\Omega) \subset W_{2,\alpha}^{(1)}(\Omega)$; from the authors' results, [1], it follows that the function $u \in V_{2,\alpha}^{(1)}(\Omega)$ has zero trace on Γ_1 .

It will be said that a function g on $\partial\Omega$ is in the space $W_{2,\alpha}^{(1/2)}(\partial\Omega)$ if there exists a function $\tilde{g} \in W_{2,\alpha}^{(1)}(\Omega)$ such that g is the trace of $\tilde{g}$ on $\partial\Omega$. The function $\tilde{g}$ is said to be the prolongation of g in Ω , and we define

$$\|g\|_{W_{2,\alpha}^{(1/2)}(\partial\Omega)} = \inf \|\tilde{g}\|_{W_{2,\alpha}^{(1)}(\Omega)} ,$$

where the infimum is taken over all prolongations $\tilde{g}$ of the function g . We shall always consider those prolongations $\tilde{g}$ for which

$$\|\tilde{g}\|_{W_{2,\alpha}^{(1)}(\Omega)} \leq c\, \|g\|_{W_{2,\alpha}^{(1/2)}(\partial\Omega)}$$

with c some positive constant.

The space $L_{2,\varphi,\alpha}(\Gamma_2)$ is defined as the set of all functions u on Γ_2 with the finite norm

$$(2.4) \qquad \| u \|_{L_{2,\varphi,\alpha}(\Gamma_2)} = \Big(\int_{\Gamma_2} \frac{u^2}{\varphi} \rho^{\alpha} d\sigma \Big)^{1/2} .$$

The most important space for our consideration is the space

$$S^{(1)}_{2,\alpha}(\Omega) = V^{(1)}_{2,\alpha}(\Omega) \cap L_{2,\varphi,\alpha}(\Gamma_2)$$

with the norm

$$(2.5) \quad \| u \|_{S^{(1)}_{2,\alpha}(\Omega)} = \Big(\| u \|^2_{V^{(1)}_{2,\alpha}(\Omega)} + \| u \|^2_{L_{2,\varphi,\alpha}(\Gamma_2)} \Big)^{1/2} .$$

The space $L_{2,\varphi,\alpha}(\Gamma_2)$ characterizes the trace of the function $u \in S^{(1)}_{2,\alpha}(\Omega)$ on Γ_2 ; the trace of u on Γ_1 is obviously zero since $S^{(1)}_{2,\alpha}(\Omega) \subset V^{(1)}_{2,\alpha}(\Omega)$.

Remark. Let $\alpha = 0$ and $\varphi(x) \geqq c\rho(x)$, where c is a positive constant. Then

$$\int_{\Gamma_2} \frac{u^2}{\varphi} d\sigma \leqq \frac{1}{c} \int_{\Gamma_2} \frac{u^2}{\rho} d\sigma .$$

It follows from the properties of traces of functions from $W^{(1)}_2(\Omega)$ that for every $u \in W^{(1)}_2(\Omega)$ (and thus also for every $u \in V^{(1)}_2(\Omega)$) the latter integral is necessarily finite and can be estimated by the norm $\| u \|_{V^{(1)}_2(\Omega)}$. Thus in this case $S^{(1)}_2(\Omega) = V^{(1)}_2(\Omega)$.

We shall so assume that $\varphi(x) \leqq c\,\rho(x)$.

Let $\mathcal{D}(\Omega)$ be the set of all infinitely differentiable functions with compact support in Ω . Let Q be a normal space, i.e. $Q = \overline{\mathcal{D}(\Omega)}$ in the norm of the space Q , and let $Q \supset S^{(1)}_{2,\alpha}(\Omega)$ algebraically and topologically (for example $Q = L_2(\Omega)$). Let

Q' be the space of all continuous linear functionals on Q (i.e. Q' is the space dual to Q).

The space dual to $S^{(1)}_{2,\alpha}(\Omega)$ is denoted by $S^{(-1)}_{2,-\alpha}(\Omega)$; the space dual to $L_{2,\varphi,\alpha}(\Gamma_2)$ may be identified with the space $L_{2,\varphi,-\alpha}(\Gamma_2)$ in the usual manner. Finally, $W^{(-1/2)}_{2,-\alpha}(\partial\Omega)$ denotes the space dual to $W^{(1/2)}_{2,\alpha}(\partial\Omega)$.

3.

Consider the operator A in the form (1.1) and the boundary value problem (1.2) and (1.3). Assume that

1) the functions $a_{ij}(x_1, x_2)$ are measurable bounded in Ω, and the quadratic form $\sum_{i,j=1}^{2} a_{ij}(x_1,x_2)\xi_i\xi_j$ is positive definite uniformly with respect to $x=(x_1,x_2)\in\Omega$;

2) the function $c(x_1,x_2)$ is positive measurable;

3) the function $\varphi(x_1,x_2)$ (see boundary condition (1.2)) fulfils a Lipschitz condition; so *we obviously have*

$$\varphi(x_1,x_2)\leqq c\,\rho(x_1,x_2) \text{ for } (x_1,x_2)\in\partial\Omega .$$

To the operator A there corresponds the bilinear form

$$a(u,v)=\int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx+\int_\Omega c\cdot u\cdot v\,dx ; \tag{3.1}$$

from the ellipticity of A it follows that

$$|a(u,u)|\geqq c\,\|u\|^2_{W_2^{(1)}(\Omega)} .$$

To the mixed problem (1.2) and (1.3) there corresponds the bilinear form

$$B(u,v)=a(u,v)+\int_{\Gamma_2}\frac{uv}{\varphi}\,d\sigma \tag{3.2}$$

defined on the cartesian product $S^{(1)}_{2,\alpha}(\Omega) \times S^{(1)}_{2,-\alpha}(\Omega)$:
it can be easily shown that

(3.3) $|B(u,v)| \leqq c_1 \|u\|_{S^{(1)}_{2,\alpha}(\Omega)} \|v\|_{S^{(1)}_{2,-\alpha}(\Omega)}$,

(3.4) $|B(u,u)| \geqq c_2 \|u\|^2_{S^{(1)}_2(\Omega)}$ (i.e. $\alpha = 0$) .

Definition. The bilinear form $B(v,u)$ is said to be $|\alpha|$-elliptic, if there are positive constants c_3 and c_4 such that

$$\sup_{\|v\|_{S^{(1)}_{2,-\alpha}(\Omega)}=1} |B(u,v)| \geqq c_3 \|u\|_{S^{(1)}_{2,\alpha}(\Omega)} ,$$

$$\sup_{\|u\|_{S^{(1)}_{2,\alpha}(\Omega)}=1} |B(u,v)| \geqq c_4 \|v\|_{S^{(1)}_{2,-\alpha}(\Omega)} .$$

Now, we have

Theorem 1. Under the corresponding hypotheses to the form $B(u,v)$, there exists an interval $\mathcal{J} = (-\gamma_1, \gamma_2)$ ($\gamma_i > 0$) such that for $\alpha \in \mathcal{J}$ the form $B(u,v)$ is $|\alpha|$-elliptic.

Remark. If $B(u,v) = B(v,u)$, then $\gamma_1 = \gamma_2$.

Next we have, by the generalized Lax-Milgram theorem, [2], the following

Theorem 2. Let $\alpha \in \mathcal{J}$, let F be a functional on the space $S^{(1)}_{2,-\alpha}(\Omega)$ (i.e. $F \in S^{(-1)}_{2,\alpha}(\Omega)$). Then there exists precisely one element $w \in S^{(1)}_{2,\alpha}(\Omega)$ such that

$$B(w,v) = F(v)$$

for every $v \in S^{(1)}_{2,-\alpha}(\Omega)$, and that

$$\|w\|_{S^{(1)}_{2,\alpha}(\Omega)} \leqq c_5 \|F\|_{S^{(-1)}_{2,\alpha}(\Omega)} :$$

4.

From Theorem 2 we obtain the existence and uniqueness of the weak solution of the mixed problem (1.2) and (1.3); the exact formulation of this problem will be given in section 5.

In all further considerations we assume $\alpha \in \mathcal{J}$, where $\mathcal{J}$ is an interval as described by Theorem 1.

Let $f \in Q'$ and let g be a functional on the space of traces of functions from $S^{(1)}_{2,-\alpha}(\Omega)$; we assume that g can be decomposed thus:

(4.1) $$g = g_1 + g_2 + \varphi g_3 ,$$

where $g_1 \in W^{(1/2)}_{2,\alpha}(\partial\Omega)$ (with the corresponding prolongation $\tilde{g}_1 \in W^{(1)}_{2,\alpha}(\Omega)$), $g_2 \in L_{2,\varphi,\alpha}(\Gamma_2)$ and we put $g_2 = 0$ for $x \in \Gamma_1$, and $g_3 \in W^{(-1/2)}_{2,\alpha}(\partial\Omega)$; for $\psi \in S^{(1)}_{2,-\alpha}(\Omega)$, $\varphi g_3(\psi)$ means the same as $g_3(\varphi\psi)$.

Let $\psi \in S^{(1)}_{2,-\alpha}(\Omega)$; setting

(4.2) $$F(\psi) = f(\psi) - a(\tilde{g}_1, \psi) + \int_{\Gamma_2} \frac{g_2\psi}{\varphi} + g_3(\psi) ,$$

we have the

Theorem 3. The functional F from (4.2) is in the space $S^{(-1)}_{2,\alpha}(\Omega)$.

It follows from Theorem 2 that there is precisely one element $w \in S^{(1)}_{2,\alpha}(\Omega)$ such that $B(w,\psi) = F(\psi)$ for every $\psi \in S^{(1)}_{2,-\alpha}(\Omega)$. Set $u = w + g_1$, and let g_1^*, g_2^*, g_3^* be functionals which form another decomposition of the functional g from (4.1). i.e.

$$g = g_1 + g_2 + \varphi g_3 = g_1^* + g_2^* + \varphi g_3^* .$$

Let F^* be the functional corresponding to f, g_1^*, g_2^* and g_3^* in a manner similar to (4.2), and let $u^* = w^* + g_1^*$, where w^* is the solution of the equation $B(w^*, \psi) = F^*(\psi)$. Then we have

<u>Theorem 4</u>. Under the above hypotheses, $u^* = u$.

5.

<u>Definition</u>. Let f be a functional on $S_{2,-\alpha}^{(1)}(\Omega)$; let g be a functional on the space of traces of functions from $S_{2,-\alpha}^{(1)}(\Omega)$ with corresponding decomposition of the form (4.1). Let F be defined by (4.2).

The function $u \in W_{2,\alpha}^{(1)}(\Omega)$ is said to be a <u>weak solution of the mixed problem (1.2) and (1.3)</u>, if

1) $u - \tilde{g}_1 \in S_{2,\alpha}^{(1)}(\Omega)$,

2) $B(u - \tilde{g}_1, \psi) = F(\psi)$ for every $\psi \in S_{2,-\alpha}^{(1)}(\Omega)$.

<u>Theorem 5</u>. Let $\alpha \in \mathcal{J}$. Then there exists precisely one weak solution $u \in W_{2,\alpha}^{(1)}(\Omega)$ of the mixed problem (1.2) and (1.3), and the estimates

$$\| u - g_1 \|_{S_{2,\alpha}^{(1)}(\Omega)} \leq c \, \| F \|_{S_{2,\alpha}^{(-1)}(\Omega)}$$

and

$$\| u \|_{W_{2,\alpha}^{(1)}(\Omega)} \leq c \, (\| f \| + \| g_1 \| + \| g_2 \| + \| g_3 \|)$$

hold (the norms are considered in the corresponding spaces).

<u>Theorem 6</u>. If $|\alpha|$ is sufficiently small and $u \in V_{2,\alpha}^{(1)}(\Omega)$ is such that $B(u, \psi) = 0$ for every $\psi \in S_{2,-\alpha}^{(1)}(\Omega)$, then $u \equiv 0$.

This theorem extends the assertion on uniqueness of solution, proved in Theorem 5 for the space $S_{2,\alpha}^{(1)}(\Omega)$, to the larger space $V_{2,\alpha}^{(1)}(\Omega)$

6.

In this section it is established that the weak solution, defined in section 5, solves the problem (1.2) and (1.3) in the classical sense, if every element is a sufficiently smooth function.

1. The condition $u - g_1 \in S^{(1)}_{2,\alpha}(\Omega)$ yields $u = g_1 = g$ on Γ_1.

2. We shall consider functions ψ which are zero in the neighbourhood of Γ_1; the equality $B(u - \tilde{g}_1, \psi) = F(\psi)$ can then be rewritten as

$$B(u,\psi) = B(\tilde{g}_1,\psi) + F(\psi) = a(\tilde{g}_1,\psi) + \int_{\Gamma_2} \frac{g_1\psi}{\varphi}\, d\sigma +$$

$$+ f(\psi) - a(\tilde{g}_1,\psi) + \int_{\Gamma_2} \frac{g_2\psi}{\varphi}\, d\sigma + \int_{\Gamma_3} g_3 \psi \, d\sigma =$$

$$= f(\psi) + \int_{\Gamma_2} \frac{g\psi}{\varphi}\, d\sigma ,$$

i.e.

$$a(u,\psi) + \int_{\Gamma_2} \frac{u\psi}{\varphi}\, d\sigma = \int_{\Omega} f\psi \, dx + \int_{\Gamma_2} \frac{g\psi}{\varphi}\, d\sigma .$$

By Green's theorem,

$$a(u,\psi) = \int_{\Omega} Au \cdot \psi \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \psi \, d\sigma ,$$

i.e.

$$\int_{\Omega} Au \cdot \psi \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \psi \, d\sigma = \int_{\Omega} f\psi \, dx + \int_{\Gamma_2} \frac{(g-u)\psi}{\varphi}\, d\sigma .$$

For $\psi \in \mathcal{D}(\Omega)$ we have $\int_{\Omega} Au \cdot \psi dx = \int_{\Omega} f\psi \, dx$ and thus

$$Au = f \quad \text{in} \quad \Omega .$$

But in this case we have for $\psi \not\equiv 0$ on Γ_2

$$\int_{\Gamma_2} \frac{\partial u}{\partial \nu} \psi \, d\sigma = \int_{\Gamma_2} \frac{g-u}{\varphi} \psi \, d\sigma$$

and thus $\frac{\partial u}{\partial \nu} = \frac{g-u}{\varphi}$ on Γ_2 ; therefore

$$u + \varphi \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma_2 ,$$

establishing that our formulation is meaningful.

R e f e r e n c e s :

[1] A. KUFNER, J. KADLEC: Characterization of functions with zero traces by integrals with weight functions, to appear in Časopis pro pěstování matematiky, 1966.

[2] J. NEČAS: Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationelle, Ann.Scuola Nor. Sup.Pisa,ser.3,16,4(1962),305-326.

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