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ON THE MINIMAX PRINCIPLE FOR K-POSITIVE OPERATORS

(Preliminary communication)

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The purpose of this note is a generalization of the well known Frobenius theorem on matrices with non negative elements, and in particular of the corresponding minimax principle.

The definitions and propositions will only be formulated here; full proofs will appear in [2].

We shall investigate a linear bounded operator T on a real Banach space Y with a closed cone K. As usual this cone induces an ordering of Y, defined by letting $x \prec y$ iff $y - x \in K$. It will be assumed that K has the following two properties:

(α) Every $x \in Y$ can be expressed in the form $x = x_1 - x_2$, where $x_1, x_2 \in K$;

(β) $\| x + y \| \geq \| x \|$ for $x, y \in K$.

The space dual to Y will be denoted by Y', and the space of continuous linear mappings of Y into itself by $[Y]$.

An operator $T \in [Y]$ is called K-<u>positive</u>, if $x \in K$ implies $Tx = y \in K$; u_0-<u>positive</u>, if it is K-positive and there is a vector $u_0 \in K$, $\| u_0 \| = 1$, such that for every $x \in K$, $x \neq o$, there exist positive numbers $\alpha = \alpha(x)$, $\beta = \beta(x)$ and a positive integer $p = p(x)$ with

$$\alpha u_0 \prec T^p x \prec \beta u_0 \quad ;$$

uniformly u_0-positive, if it is u_0-positive and the positive integer p does not depend on x.

The value of a form $x' \in Y'$ at $x \in Y$ will be denoted by $\langle x, x' \rangle$.

A set $H' \subset K'$, where K' is the cone adjoint to K, is called K-total, if $\langle x, x' \rangle \geqq 0$ for all $x' \in H'$ implies $x \in K$.

Theorem 1. Under the assumptions

(i) $K \subset Y$ has properties (α) and (β);

(ii) $H' \subset K'$ is a K-total set;

(iii) T is a u_0-positive operator;

(iv) There is only a finite number of singularities $\mu_1, \ldots, \mu_s$ of the resolvent $\mathcal{R}(\lambda, T) = (\lambda I - T)^{-1}$, for which $|\mu_j| = r(T)$, where $r(T)$ is the spectral radius of T.

Moreover let all $\mu_1, \ldots, \mu_s$ be poles of $\mathcal{R}(\lambda, T)$; then

1. $$\mu_1 = r(T) = \min_{\substack{x \in K \\ x \neq 0}} \sup_{x' \in H'} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle} = \max_{x \in K} \inf_{\substack{x' \in H' \\ \langle u_0, x' \rangle \cdot \langle x, x' \rangle > 0}} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle} \; ;$$

2. The point μ_1 is a proper value of T and to it there corresponds a u_0-positive proper vector x_0. Every proper vector $x \in K$ of the operator T has the form $x = cx_0$, where $c > 0$.

The vector $\tilde{x} \in K$ is called extremal with respect to

T , if

$$\operatorname*{Max}_{x \in K} r_x = r_{\tilde{x}} \qquad \text{or} \qquad \operatorname*{Min}_{\substack{x \in K \\ x \neq 0}} r^x = r^{\tilde{x}} ,$$

where

$$r_x = \inf_{\substack{x' \in H' \\ \langle u_0, x' \rangle \cdot \langle x, x' \rangle > 0}} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle} ,$$

and

$$r^x = \sup_{x' \in H'} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle} .$$

Theorem 2. Let the assumptions of Theorem 1 be fulfilled. Moreover let T be a uniformly u_0-positive operator. Then every vector extremal with respect to T has the form cx_0 , where x_0 ($\| x_0 \| = 1$) is the unique proper vector of T lying in K .

The applications of these theorems are similar to those of the Frobenius theorem. For example, one can obtain the infinite-dimensional analogue of the Stein-Rosenberg theorem [1, p. 105], also some theorems on localization of spectra, and other related results. Even in the finite-dimensional case, Theorems 1 and 2 are slightly more general than the known theorems, since a u_0-positive matrix need not be necessarily irreducible.

R e f e r e n c e s :

[1] A.S. HOUSEHOLDER: The Theory of Matrices in Numerical Analysis. Blaisdell Publ.Comp. New York 1964.

[2] I. MAREK: Some spectral properties of K-positive operators and inclusion theorems for the spectral radius.(To appear in Czech.Math. Journ.)

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