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A note on Chehata's groups

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Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
1. The purpose of this note is to establish the existence of a class of maximal subgroups in the simple groups $G(F)$ and $G(J')$.

Terminology, notation, and fundamental facts about these groups to be used later are given in [1].

2. Following Chehata [1], we denote by $\Phi(J), J=\alpha, \beta]$, the subset of $G(J)$ of all elements which have three breaks only, by $S(J)$ the subset of $\Phi(J)$ of elements with three breaks, the first at $\alpha$, the second between $\alpha$ and $\beta$, and the third at $\beta$.

Let $J'=\alpha, \beta$, $\alpha < \beta$, be an open interval, $\alpha, \beta \in F$, $\xi_0 \in (\alpha, \beta)$. The set of all elements $f \in G(J')$ such that $f(\xi_0)=\xi_0$ we denote by $W(\xi_0, J')$. Similarly is defined $W(\xi_0, F)$.

3. The following assertions will be used in the proof of the main theorems:

Lemma (3,1). Let $\alpha < \gamma < \xi < \sigma < \beta$ and let $f$ be an element of $G((\alpha, \beta))$ which has the break at $\xi$, but in $(\gamma, \xi)$ and $(\xi, \sigma)$ it has no breaks.

Then there is an element $k \in S(\gamma, \sigma)$ such that

(i) $f = fk$

(ii) $f(\xi)=f(\xi)$ if $f \notin (\gamma, \sigma)$

(iii) $f$ has no breaks in $(\gamma, \sigma)$.
This Lemma is a generalization of Chehata's Lemma 5 and can be proved by similar methods.

Corollary (3,2). Let \( \alpha < \xi_0 < \beta_1, \gamma' = (\alpha, \beta) \) and let \( f \) be an element of \( G(\gamma') = M(\xi_0, \gamma') \),

\[
f = \{ \xi_1^A \xi_1 \eta_1 \ldots \xi_{n-2} \eta_{n-1} \}
\]

Then there are \( g_i \in \xi (\xi_0, \mu), i = 1, \ldots, \beta \) and

\( h_j \in \xi (\xi_0, \mu), j = 1, \ldots, \beta, \gamma_0 + t < \gamma - 2 \) such that

(i) \( f^* = g_1 \ldots g_n h_1 \ldots h_\gamma \)

(ii) \( f^* (\xi_0) = f(\xi_0) \)

(iii) \( f^* (\xi_0) = \xi \) for \( \xi \in (\lambda', \mu') \), \( \lambda \leq \lambda' < \xi_0 < \mu' \leq \mu \)

(iv) \( f^* \in S (\gamma', \mu') \).

Corollary (3,3). If we assume in Lemma (3,1) that moreover \( f(\xi) > \xi \) for all \( \xi \in (\gamma', \sigma') \), then \( f(\xi) > \xi \) for \( \xi \in (\gamma, \sigma') \).

Lemma (3,4). Let \( f^* \in S (\gamma, \sigma) = M(\xi_0, (\alpha, \beta)), \alpha < \gamma < \xi_0 < \sigma < \beta \), be an element of the form

\[
f^* = \{ \xi_0 \xi_1 \xi_2 \sigma' 1 \}
\]

and let \( f^* (\xi_0) > \xi_0 \).

If \( \rho \in S, \xi_0 < \rho < \sigma \), then there are elements \( \ell_1, \ell_2 \in M(\xi_0, (\alpha, \beta)) \) which have the following properties:

1) \( f' = \ell_1 f^* \ell_2 \in S (\gamma, \sigma) \)

2) \( f' (\xi_0) = \rho \).

Proof: If we take

\[
\ell_1 = \{ \xi_0 \xi_0' \xi_0^+ f^* (\xi_0), f^* (\xi_0), \sigma' 1 \}
\]

and choose \( \xi_0', \psi \) such that

\( \ell_1 (f^* (\xi_0)) = \rho \),

then

- 118 -
\[ l_1 f^+ = \{ \gamma \omega_1 f^+ \tau(\xi_0) \mid \omega \in \Omega, \xi_0 \in \Sigma \} \]

This element we can transform by a suitable \( l_2 \) to an element which has its break in \((\gamma, \sigma')\) at \( f^+ (\xi_0) \) only; this is possible by Lemma (3,1). Hence we obtain

\[ f' = l_1 f^+ l_2 \in \Phi (\, \gamma, \sigma') \]

and

\[ (l_1 f^+ l_2)(\xi_0) = (l_1 f^+)(l_2(\xi_0)) = (l_1 f^+)(\xi_0) = \Phi \cdot \]

Since

\[ (l_1 f^+)(\xi_0) = \xi_0 > f^+ \tau(\xi_0) \]

and

\[ (l_1 f^+)(\xi_0) = \Phi > \xi_0 \cdot \]

it follows that

\[ (l_1 f^+)(\xi) > \xi \text{ if } \xi \in (\gamma, \sigma') \cdot \]

Consequently, by Corollary (3,3)

\[ f'(\xi) > \xi \text{ if } \xi \in (\gamma, \sigma') \]

and we conclude

\[ f' \in S (\, \gamma, \sigma') \cdot \]

Lemma (3,5). If \( \alpha < \gamma' < \xi_0 < \sigma' < \beta, \alpha < \gamma < \xi_0 < \sigma' < \beta \), and \( \gamma \in G (\, \gamma', \sigma') \), then there is an element \( m \in M (\xi_0, (\alpha, \beta)) \) such that \( m \gamma m^{-1} \in G (\, \gamma', \sigma') \).

The proof is analogous to that of Lemma 1 in [1] and is here omitted.

4. The main results are expressed in the following theorems:

Theorem (4,1). The group \( M (\xi_0, (\alpha, \beta)), \xi_0 \in (\alpha, \beta) \), is a maximal subgroup of \( G ((\alpha, \beta)) \).
Proof: Obviously $M(\xi, (\alpha, \beta)) \in G((\alpha, \beta))$. Suppose $f \notin M(\xi, (\alpha, \beta))$. I.e. $f(\xi) > \xi$. We can assume without loss of generality that $f(\xi) > \xi$. Using Corollary (3,2), we transform $f$ to the element $f^*$ and since $f^*(\xi) = f(\xi) > \xi$ it implies $f^* \in M((\alpha, \beta)) - M(\xi, (\alpha, \beta))$. Let us denote by $\mathcal{N} = \{M(\xi, (\alpha, \beta)), f\}$ the subgroup generated by elements of $M(\xi, (\alpha, \beta))$ and $f$. Then $f^* \in \mathcal{N}$ and by Lemmas (3,4) and (3,1) $f' \in \mathcal{N}$ if $f' \in S([\gamma, \sigma])$ and $\xi_0 < f'(\xi_0) = \varphi < \sigma$, where $\gamma = \lambda(f^*)$, $\sigma = \mu(f^*)$.

Now it is easy to show that $\mathcal{N} \supset S([\gamma, \sigma])$. Indeed, if $f'' \in S([\gamma, \sigma])$, then either $f''(\xi_0) > \xi_0$ or $f''^{-1}(\xi_0) > \xi_0$. But if $f''(\xi_0) > \xi_0$, we know already that $f'' \in \mathcal{N}$. If $f''^{-1}(\xi_0) > \xi_0$, let us multiply this element on the right by a suitable element $\ell \in M(\xi, (\alpha, \beta)) = M$, using Corollary (3,2) and Lemma (3,4). We get $f''^{-1}\ell \in S([\gamma, \sigma])$ and since $(f''\ell)(\xi_0) = f''^{-1}(\xi_0) > \xi_0$, it follows that $f''^{-1}\ell \in \mathcal{N}$ and thus $f'' \in \mathcal{N}$. Hence $\mathcal{N} \supset \{S([\gamma, \sigma]) \} = G([\gamma, \sigma])$.

If $\varphi$ is an element of $G((\alpha, \beta))$ and $\varphi \notin \mathcal{N}$, then $\lambda(\varphi) < \xi_0$, $\mu(\varphi) > \xi_0$. Let us write $\lambda(\varphi) = \gamma'$, $\mu(\varphi) = \sigma'$. By Lemma (3,5) there is an element $m \in \mathcal{N}$ such that

\[ m \varphi m^{-1} \in G([\gamma, \sigma']) \subset \mathcal{N}. \]

This implies

\[ \varphi \in m^{-1} \mathcal{N} m \subset \mathcal{N}, \]

since

\[ m \in \mathcal{N} \subset \mathcal{N}. \]

- 120 -
But then
\[ G(\langle \alpha, \beta \rangle) \subset \mathcal{K} = \{ \mathcal{M}, f \} \subset G(\langle \alpha, \beta \rangle), \]
hence
\[ G(\langle \alpha, \beta \rangle) = \{ \mathcal{M}, f \}. \]

The proof is complete.

Using Theorem (4.1), we infer the

Theorem (4.2). The group \( \mathcal{M}(\gamma, F) \), \( \gamma \in F \) is a maximal subgroup of \( G(F) \).

Bibliography:

[1] C.G. CHEHATA: An algebraically simple ordered group,

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