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AN EXAMPLE CONCERNING SMALL CHANGES OF COMMUTING FUNCTIONS

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A number of papers has been devoted in the last years to the study of pairs of commuting functions (by a function is meant, throughout this remark, a continuous transformation of the segment $[0,1]$ into itself). The aim of this remark is to give an example of extremely "discontinuous" behavior of two commuting functions: a slight modification of one may cause an unexpectedly great change of the other in order to preserve commutativity.

Given an arbitrary $\varepsilon \in (0, \frac{1}{3})$, there are constructed three piece-wise linear functions f, f^*, g such that $g \circ f = f \circ g$, $\rho(f^*, f) \leq \varepsilon$ in the uniform metric, and that $\rho(g^*, g) \geq \frac{1}{3}$ whenever $g^* \circ f^* = f^* \circ g^*$.

The function f^* also has another property. It has two fixed points 0 and $\frac{2}{3}$ and, whenever $g^* \circ f^* = f^* \circ g^*$, either $g^*(\frac{2}{3}) = \frac{2}{3}$, or g^* is identically zero.

Define the functions f and g by:

$$f(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ -2(x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases} \qquad g(x) = \begin{cases} 3x & \text{for } x \in [0, \frac{1}{3}] \\ -3x+2 & \text{for } x \in [\frac{1}{3}, \frac{2}{3}] \\ 3x-2 & \text{for } x \in [\frac{2}{3}, 1]. \end{cases}$$

Clearly, f and g are continuous and commute under composition.

Now, we shall modify the function f in a small neighborhood of its fixed point $\frac{2}{3}$, putting for $0 < \varepsilon < \frac{1}{3}$

$$f^*(x) = \begin{cases} f(x) & \text{for } x \in [0, \frac{1}{3} - \frac{\varepsilon}{2}] \cup [\frac{1}{3} + \frac{\varepsilon}{2}, 1] \\ -3x + \frac{8}{3} - \frac{\varepsilon}{2} & \text{for } x \in [\frac{1}{3} - \frac{\varepsilon}{2}, \frac{1}{3} - \frac{\varepsilon}{4}] \\ -x + \frac{4}{3} & \text{for } x \in [\frac{1}{3} - \frac{\varepsilon}{4}, \frac{\varepsilon}{2} + \frac{\varepsilon}{4}] \\ -3x + \frac{8}{3} + \frac{\varepsilon}{2} & \text{for } x \in [\frac{1}{3} + \frac{\varepsilon}{4}, \frac{2}{3} + \frac{\varepsilon}{2}] . \end{cases}$$

There is $\rho(f^*, f) \leq \frac{\varepsilon}{4}$ in the uniform metric ρ, and we shall prove that for any continuous function g^* commuting with f^* the inequality $\rho(g^*, g) \geq \frac{1}{3}$ holds.

Define an equivalence E on $[0, 1]$ by $x \, E \, y$ if and only if $f^{*m}(x) = f^{*n}(y)$ for some positive integers m, n. The set $E[x]$ of elements equivalent with x will be called the component of x. The usefulness of such an equivalence is based on the fact that if $x \, E \, y$ then also $g^*(x) \, E \, g^*(y)$ for any function g^* commuting with f^*.

Put $X_0 = [\frac{1}{3} - \frac{\varepsilon}{4}, \frac{1}{3} + \frac{\varepsilon}{4}]$. First show that $E[X_0] = \bigcup_{x \in X_0} E[x]$ is dense in $[0, 1]$.

Let U be an open interval of length η, and suppose $f^{*n}(U) \cap X_0 = \emptyset$ and $\frac{1}{2} \notin f^{*n}(U)$ for $n = 0, 1, \dots$. Since the slope of f^* is not less than 2 on $[0,1] \setminus X_0$ and no $f^{*n}(U)$ contains the point $\frac{1}{2}$, the length of $f^{*n}(U)$ increases geometrically with n, contrary to $f^{*n}(U) \subset [0, 1]$. Therefore, for some n_0 we have either $\frac{1}{2} \in f^{*n_0}(U)$ or $f^{*n_0}(U) \cap X_0 = Y \neq \emptyset$.

In the case $Y \neq \emptyset$ for some $\xi \in f^{*(-n_0)}(Y) \cap U \neq \emptyset$, there is $f^{*n_0}(\xi) \in X$, i.e. $\xi \in E[X_0]$.

If $f^{*n_0}(U)$ contains $\frac{1}{2}$, put $\xi = \frac{1}{2} - \frac{1}{3.2^n}$ taking n such that $\xi \in f^{*n_0}(U)$; then $f^{*(n+1)}(\xi) = \frac{2}{3} \in X_0$.

It is also easily seen that $E[0] \cap E[X_0] = \emptyset$, since if $x \in E[0]$ then $f^{*n}(x) = 0$ for sufficiently large n, whereas $f^{*n}(y) \in X_0$ for $y \in E[X_0]$ and large n.

Now let g^* be a continuous function commuting with f^*. The set $\{0, \frac{2}{3}\}$ consisting of fixed points of the function f^* is invariant under g^*, therefore $g^*(\frac{2}{3}) = \frac{2}{3}$ or $g^*(\frac{2}{3}) = 0$. In the first case $\rho(g^*, g) \geq \frac{2}{3}$, since $g(\frac{2}{3}) = 0$.

Assume $g^*(\frac{2}{3}) = 0$. We are going to show that $g^*(x) = 0$ for every $x \in [0, 1]$.

First, a couple (x, y), $x, y \in [0, 1]$, $x \neq y$, is called a 2-cycle of f^* if $f^*(x) = y$, $f^*(y) = x$. Evidently, the image of a 2-cycle under g^* is a 2-cycle or a fixed point of f^*. Observe that any point in X_0 is either fixed or belongs to a 2-cycle. As g^* is continuous, the segment X_0 must be mapped onto a segment containing 0, every point of which is either fixed or belongs to a 2-cycle. By definition of f^*, there is no proper segment with this property. Hence, $g^*(X_0) = 0$.

But this means that $E[X_0]$ is carried by g^* into $E[0]$. It is easy to see that $E[0]$ is nowhere dense. Indeed, since $E[X_0]$ is the union of closed non-trivial intervals $\bigcup_{n=0}^{\infty} f^{*(-n)}(X_0)$ and is dense in $[0, 1]$, its complement is nowhere dense. It follows that $E[X_0]$ is mapped by g^* onto 0. As $E[X_0]$ is dense and g^* continuous, there is $g^*(x) = 0$ for every $x \in [0, 1]$.

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