Otomar Hájek

Differentiable representation of flows (Preliminary communication)

Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 2, 219--225

Persistent URL: http://dml.cz/dmlcz/105054

Commentationes Mathematicae Universitatis Carolinae
7, 2 (1966)

DIFFERENTIABLE REPRESENTATION OF FLOWS
(Preliminary communication)

Otomar HÁJEK, Praha

The main result, Theorem 6, will be published later with full proofs, in [2].

1. A (bilateral local) <u>flow</u> in P is a partial map $t: R^1 \times P \times R^1 \to P$ with properties 1° to 3° below. To express these reasonably, it is useful to introduce the system of movements $\{{}_\alpha t_\beta \mid \alpha, \beta \text{ in } R^1\}$ of t; an individual movement ${}_\alpha t_\beta$ is then defined as the partial map ${}_\alpha t_\beta : P \to P$ such that

$${}_\alpha t_\beta x = t(\alpha, x, \beta)$$

iff the right side is defined. The three requirements are then

1° ${}_\alpha t_\alpha x = x$ whenever the left side is defined,

2° For all $\alpha \geq \beta \geq \gamma$ in R^1 there is

$${}_\alpha t_\beta \circ {}_\beta t_\gamma = {}_\alpha t_\gamma \quad , \tag{1}$$

3° If ${}_\alpha t_\alpha x$ is defined, then so is ${}_\theta t_\alpha x$ for sufficiently small $|\theta - \alpha|$.

These may be termed, in turn, the initial value property, the compositivity property, and the local existence condition. In 2°, (1) should be interpreted strictly, with composition of partial maps; in particular, if ${}_\alpha t_\gamma x$ is defined and β is taken arbitrarily with $\alpha \geq \beta \geq \gamma$, then ${}_{\beta}t_{\gamma}x$ and ${}_{\alpha}t_{\beta}\,({}_{\beta}t_{\gamma}x)$ must both be defined.

2. The significant subset of $P \times R^1$,

$$\mathfrak{D} = \{(x, \alpha) : {}_{\alpha}t_{\alpha}x \quad \text{defined}\}$$

will be termed the <u>solution-space</u> of t. The flow t will be called <u>global</u> iff 1.3° may be strengthened to conclude: ... for all $\theta \in R^1$. The flow t will be called <u>immobile</u> iff 1.1° may be strengthened to: ${}_{\theta}t_{\alpha}x = x$ whenever defined.

3. Assume given a flow t in P and also a topology on P; then t will be termed a <u>continuous</u> flow in the topological space P iff the following further conditions are satisfied:

1° domain t is open in $R^1 \times P \times R^1$,

2° t : domain $t \rightarrow P$ is continuous.

(Here and later, the given topology on P and the natural topology on R^1 induce product topologies on $P \times R^1$, $R^1 \times P \times R^1$, etc., and then also a subset topology on $\mathfrak{D} \subset P \times R^1$, etc.; these are the topologies used in 1° and 2°.)

4. Assume given a differential equation in euclidean n-space R^n

$$\frac{dx}{d\theta} = f(x, \theta), \qquad (2)$$

with $f : \mathfrak{D} \rightarrow R^n$ continuous and $\mathfrak{D} \subset R^{n+1}$ open. If (indeed, if and only if) the solutions of (2) exhibit unicity (of the initial-value problem, in the usual sense), then (2) defines a flow t in R^n; this is described as follows: set $u = t(\alpha, x, \beta)$ iff $(x, \beta) \in \mathcal{D}$ and there exists a solution s of (2) with $s\beta = x$, $s\alpha = u$.

For definiteness, s is a (classical) solution of (2) iff s is a partial map $R^1 \to R^n$ with domain s an interval in R^1 and

$$\frac{d}{d\theta} s\theta = f(s\theta, \theta) \quad \text{for all } \theta \in \text{domain } s .$$

That t satisfies 1.1° to 3° is verified easily; 3.1° and 2° follow readily from classical theorems (e.g. 3.2° is a re-formulation of continuous dependence of solutions on initial data). Obviously the flow t associated with (2) describes the equation completely; thus ${}_\theta t_\alpha x$ with fixed $(x, \alpha) \in \mathcal{D}$ and variable θ is a solution of (2). Easily, the solution space of t is precisely the domain $\mathcal{D}$ of f.

Flows obtained in this manner (with the exhibited requirements on f and $\mathcal{D}$) will be termed differential. This term will also be applied to flows similarly associated with differential equations on differentiable n-manifolds (i.e. topological n-manifolds with a specified differential structure); the detailed formulation is possibly obvious.

Evidently t is global iff (2) has prolongability (i.e. global existence) of solutions; and t is immobile iff (2) is precisely $dx/d\theta = 0$.

5. It is natural to ask whether also conversely every continuous flow on a manifold is differential. Since the

flow axioms 1.1° to 3°, 2.1° and 2° involve only the topological structure but not the differential, the question must be amended to read: Is every continuous flow on a manifold homeomorphic to a differential flow? Of course this requires a definition of the concept of homeomorphic flows.

Thus, let t and t' be continuous flows on topological spaces P and P' (with solution spaces D, D' respectively); then a partial map $h : P \times R^1 \to P' \times R^1$ will be said to be a <u>homeomorphism</u> $t \to t'$ iff the following three conditions are satisfied:

1° $h : D \approx D'$ is a homeomorphism,

2° $h(x, \theta) = (h_1(x, \theta), \theta)$ for all $(x, \theta) \in D$,

3° $h_1({}_\theta t_\alpha x, \theta) = {}_\theta t'_\alpha h_1(x, \alpha)$, whenever either side is defined.

Here 1° includes the requirement that $D \subset$ domain h. Condition 2° may be loosely expressed as stating that h leaves time invariant; together with 3° this is the abstract counterpart to "transformations" of (2) of the form $y = h(x, \theta)$ with y as "new independent variable", leading to $dy/d\theta = \ldots$. Obviously then also h^{-1} is a homeomorphism $t' \to t$.

6. Having established the terminology, one may formulate the main result as follows:

<u>THEOREM</u>. To every continuous flow t on a differentiable manifold there exists a differentiable flow t' homeomorphic to t ; furthermore, t' may be taken immobile (i.e. corresponding to $dx/d\theta = 0$ in some local coordinate x).

7. This result shows that the concept of a continuous flow is a perfectly adequate generalization of differential equations(2) on manifolds; axioms 1.1° to 3° then formulate the "abstract dynamical properties", and axioms 3.1° and 2° the conditions of compatibility with the topological structure.

A continuous flow t may be interpreted directly, as describing the changes in time of a deterministic physical system; the individual movements ${}_{\alpha}t_{\beta}$ are the (actual or abstract) movements of the system. Theorem 6 may then be formulated, inaccurately but more vividly, as stating that an equivalent interpretation of t is as the observation of a completely immobile system from a time-variable point of view (this latter concerns the possible dependence on "time" θ of $h_1(x, \theta)$ in 5.2°).

8. An immediate consequence of Theorem 6 is the

COROLLARY. Let t be a continuous flow on an n-manifold. Then each point of the solution space of t is contained in some open integral set D_0 (i.e. such that $(x, \alpha) \in D_0$ implies $({}_{\theta}t_{\alpha} x, \theta) \in D_0$) with the following property: There exist n functions $h_k : D_0 \to R^1$ such that

1° each h_k is constant along trajectories of t in D_0, i.e. $h({}_{\theta}t_{\alpha} x, \theta) = h(x, \alpha)$ if $(x, \alpha) \in D_0$ and ${}_{\theta}t_{\alpha}x$ is defined, and

2° the map $h : D_0 \to R^{n+1}$ with $h(x, \theta) = (\{h_k(x, \theta)\}_{k=1}^{n}, \theta)$ is a homeomorphism of D_0 onto an open set in R^{n+1}.

9. There is a familiar connection (e.g. [3], chap. X, § 32, Satz 1) between a differential equation (2) in R^n, i.e. a system

$$\frac{d\xi_k}{d\theta} = \varphi_k(x, \theta), \qquad (k = 1, \dots, n),$$

(with $x = \{\xi_k\}_{k=1}^n$, $f = \{\varphi_k\}_{k=1}^n$, etc.), and the partial differential equation for η

$$\sum_{k=1}^{n} \varphi_k(x, \theta) \frac{\partial \eta(x, \theta)}{\partial \xi_k} + \frac{\partial \eta(x, \theta)}{\partial \theta} = 0 \ . \qquad (3)$$

Corollary 8 is then the abstract counterpart to the local existence theorem for Hauptsysteme von Integralen of (3).

10. As a matter of fact, Theorem 6 holds even for P a general topological space. The resulting immobile flow t' can be extended to a global flow t^* in the obvious manner, ${}_{\theta\alpha}t^* x = x$ for all $(x, \alpha) \in D'$ and $\theta \in R^1$. Thus one obtains a global extension procedure for flows, and this may be exhibited as the action of a faithful functor between the appropriate categories (of continuous flows and its full subcategory of continuous global flows). Actually, this last result was obtained first, in the attempt to generalize the results on global extensions of dynamical systems [1] to non-stationary flows.

References

[1] O. HÁJEK: Structure of dynamical systems, this journal, 6,1(1965),53-72.
Correction, this journal, 6,2(1965),211-212.

[2] O. HÁJEK: Process theory I - V, to appear in the Czechoslovak Mathematical Journal;

[3] E. KAMKE: Differenzialgleichungen reeler Funktionen, Teubner, Leipzig, 1930.

(Received March 21,1966)