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COORDINATIZATION OF PARALLEL SYSTEMS

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In the present note we shall investigate coordinatizing system to certain types of André's parallel systems.

Definition 1. A parallel system(1) is a triplet $\mathcal{P} = (\mathfrak{P}, \mathcal{L}, \parallel)$ where 1) $\mathfrak{P}$ is a nonvoid set of elements called points, 2) $\mathcal{L}$ is a nonvoid set of some nonvoid subsets in $\mathfrak{P}$ called lines, 3) $\parallel$ is a partition (2) of $\mathcal{L}$ such that each $\mathcal{L} \in \parallel$ is a partition of $\mathfrak{P}$, and 4) there are three points not on the same line.

$\mathcal{P}$ is called special if $\mathfrak{P} = X \times Y$ for some sets X, Y and if

(1) $\mathcal{X} = \{\{(x,y) \mid y \in Y\} \mid x \in X\} \in \parallel$, $\mathcal{Y} = \{\{(x,y) \mid x \in X\} \mid y \in Y\} \in \parallel$,

(2_1) $card\,(A \cap B) = 1$ for $A \in \parallel \setminus \mathcal{Y}$, $B \in \mathcal{Y}$.

For further application we shall formulate three further conditions:

(2_2) $card\,(A \cap B) = 1$ for $A \in \parallel \setminus \mathcal{X}$, $B \in \mathcal{X}$,

(3) if A, B are distinct points then there is exactly one line containing both A, B,

(4) $card\,(A \cap B) = 1$ for lines A, B belonging to distinct elements of $\parallel$.

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(1) Cf. [1], p.90.

(2) A partition of a set $S \neq \emptyset$ is a decomposition of S into pairwise disjoint nonvoid subsets in S covering S.

Definition 2. A **ternar** is a quintuplet $\mathcal{T} = (X, Y, U, V, T)$ where 1) X, Y, U, V are nonvoid sets with $\operatorname{card} X \geq 2$, $\operatorname{card} Y \geq 2$ and 2) T is a map of $X \times Y \times U$ onto V. Set $L(u,v) = \{(x,y) \mid T(x,y,u) = v\}$, $\mathcal{J} = \{(u,v) \mid L(u,v) \neq \emptyset\}$, $\mathcal{L} = \{L(u,v) \mid (u,v) \in \mathcal{J}\}$, $\mathcal{L}_u$ being the set of all $L(u,v) \neq \emptyset$ with fixed $u \in U$ and $\parallel = \{\mathcal{L}_u \mid u \in U\}$. $\mathcal{T}$ is called **special** if

(5) the map $(u,v) \to L(u,v)$ is a bijection of $\mathcal{J}$ onto $\mathcal{L}$,

(6) there exist elements $\sigma, \infty \in U$ and injections $\xi : X \to V$, $\eta : Y \to V$ such that $T(x,y,\sigma) = \eta y$ and $T(x,y,\infty) = \xi x$ for all $x \in X$, $y \in Y$,

(7_1) $T(x,y,u) = v$ is uniquely solvable in $y \in Y$ for given $x \in X$, $(u,v) \in \mathcal{J}$.

Next we formulate some further conditions:

(7_2) $T(x,y,u) = v$ is uniquely solvable in $x \in X$ for given $y \in Y$, $(u,v) \in \mathcal{J}$,

(8) if (x_1,y_1), (x_2,y_2) are distinct elements in $X \times Y$, then there is exactly one $u \in U$ satisfying $T(x_1,y_1,u) = T(x_2,y_2,u)$,

(9) if (u_1,v_1), (u_2,v_2) are distincts elements in $\mathcal{J}$, then the equations $T(x,y,u_1) = v_1$, $T(x,y,u_2) = v_2$ have a unique solution $(x,y) \in X \times Y$.

Proposition 1. Let $\mathcal{T} = (X, Y, U, V, T)$ be a ternar. Then $\parallel$ (cf. Definition 2) is a partition of $\mathcal{L}$ (cf. Definition 2) iff (5) holds.

Proof. If $L(u_1, v_1) = L(u_2, v_2)$ for some $(u_1, v_1), (u_2, v_2) \in \mathcal{J}$ with $u_1 \neq u_2$ then $(u, v) \to L(u, v)$ is not a bijection of $\mathcal{J}$ onto $\mathcal{L}$. Conversely, if $(u, v) \to$ $\to L(u, v)$ is a bijection of $\mathcal{J}$ onto $\mathcal{L}$, then $L(u_1, v_1) \neq$ $\neq L(u_2, v_2)$ for distinct $(u_1, v_1), (u_2, v_2) \in \mathcal{J}$.

Proposition 2. Let $\mathcal{T} = (X, Y, U, V, T)$ be a special ternar. Then $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel)$ (cf. Definition 2) is a special parallel system, to be termed associated with $\mathcal{T}$.

Proof. By Proposition 1, $\parallel$ must be a partition of $\mathcal{L}$, and by the definition of $\mathcal{L}_u$ (cf. Definition 2), each $\mathcal{L}_u$, $u \in U$, is a partition of $X \times Y$. From (6) and (7_1) there follow (1) and (2_1). Finally, from $\operatorname{card} X \geq 2$, $\operatorname{card} Y \geq 2$, and by (5), (6), (7), there exist at least four elements of $X \times Y$ which are not on the same line. Thus $\mathcal{P}$ is a special parallel system. Note that for $\mathcal{T}$ and $\mathcal{P}$, $(2_2) \Leftrightarrow (7_2)$, $(3) \Leftrightarrow (8)$ and $(4) \Leftrightarrow (9)$.

Proposition 3. Let there be given a special parallel system $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel)$, $\mathcal{L} = \{\mathcal{L}_u\}_{u \in U}$. Choose a set V such that there exist injections $\varkappa_u : \mathcal{L}_u \to V$ for all $u \in U$ and $V = \bigcup_{u \in U} \varkappa_u \mathcal{L}_u$. Define the map T: $X \times Y \times U \to V$ by $T(x, y, u) = v \Leftrightarrow (x, y) = \varkappa_u^{-1} v$. Then the ternar $\mathcal{T} = (X, Y, U, V, T)$ is special (and will be called associated with $\mathcal{P}$).

Proof. As $\parallel$ is a partition of $\mathcal{L}$, (5) holds by

Proposition 1. Furthermore, $(1) \Rightarrow (6)$ and $(2_1) \Rightarrow (7_1)$. From the fact that there are at least three points which are not on the same line, it follows that $card\ X \geq 2$, $card\ Y \geq 2$. Thus $\mathcal{T}$ is a special ternar. Note that, for $\mathcal{P}$ and $\mathcal{T}$, $(2_2) \Rightarrow (7_2)$, $(3) \Leftrightarrow (8)$ and $(4) \Leftrightarrow (9)$.

<u>Proposition 4</u>. Let $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel)$, $\mathcal{L} = \{\mathcal{L}_u\}_{u \in U}$ be a special parallel system. Then an associated special ternar $\mathcal{T} = (X, Y, U, V, T)$ can be chosen such that (using the notation of Definition 2)

(10) $o \in X \subseteq Y = V \supseteq U \setminus \{\infty\}$,

(11) $T(x, y, o) = y$, $T(x, y, \infty) = x$ for all $x \in X, y \in Y$,

(12) $T(o, v, u) = v$ for all $u \in U$, $v \in V$,

(13) there is an element $e \in X \setminus \{o\}$ satisfying $T(x, x, e) = o$, $T(e, u, u) = o$ for all $x \in X$, $u \in U \setminus \{\infty\}$.

<u>Proof</u>. a) Choose a point $O = (o_1, o_2)$ and a line $\{(x, y) \mid x = e\}$ with $e \neq o_1$. Let ρ be the injection of $U \setminus \{\infty\}$ into Y defined as follows: For each $u \in U \setminus \{\infty\}$, let $(e, \rho u) \in L$, where $O \in L \in \mathcal{L}_u$. Thus we can identify each $u \in U \setminus \{\infty\}$ with ρu, and obtain $U \setminus \{\infty\} \subseteq Y$.

b) Choose a line E with $O \in E \in \mathcal{L}_{e_2}$ for some $e_2 \in U \setminus \{o, \infty\}$, and define an injection $\sigma: X \rightarrow Y$ by $(x, \sigma x) \in E$ for each $x \in X$. Then we can identify each $x \in X$ with σx, and obtain $X \subseteq Y$. After this identification, we have $o = o_1 = o_2$, $e = e_2$.

c) It is possible to take ξ, η as identity maps.

Then (11) is satisfied.

d) Let each line $L \in \mathcal{L}_u$, $u \in U \setminus \{\infty\}$, be uniquely determined by the "intercept"[3] $(\sigma, v) \in L$ so that $L = \{(x,y) \mid T(x,y,u) = \mathscr{K}_u L\}$. We have the bijection λ_u: $V \to Y$ where $\lambda_u(\mathscr{K}_u L) = v$ for each $L \in \mathcal{L}_u$. After identifying each $\mathscr{K}_u L$ with $\lambda_u(\mathscr{K}_u L)$, we obtain $V = Y$. Thus (10) is proved.

e) $(\sigma, v) \in \{(x,y) \mid T(x,y,u) = v\} \Rightarrow (12)$,

$E = \{(x,y) \mid T(x,y,e) = \sigma\} \Rightarrow (13_1)$ and

$(e,y) \in \{(x,y) \mid T(x,y,e) = \sigma\} \Rightarrow (13_2)$.

<u>Proposition 5</u>. Let $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel)$ be a special parallel system, and let $\mathcal{T} = (X, Y, U, Y, T)$ be the associated special ternar constructed in Proposition 4. Define two derived maps $X \times Y \to Y$ (denoted as addition) and $X \times U \to Y$ (denoted as multiplication) by

(14) $T(x, x+y, e) = y$ for $x \in X$, $y \in Y$,

(15) $T(x, x \cdot y) = \sigma$ for $x \in X$, $y \in U$.[4]

Then

(16_1) $x + \sigma = x$ for $x \in X$,

(16_2) $\sigma + y = y$ for $y \in Y$,

(17_1) $x + y = z$ is uniquely solvable in $x \in X$ for given $y \in Y$, $z \in Y$,

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3) Compare with [2], p.5 or [4], p.503, respectively.

4) Compare with [4], p.505.

(17_2) $x + y = z$ is uniquely solvable in $y \in Y$ for given $x \in X$, $z \in Y$,

(18_1) $x \cdot \sigma = x$ for $x \in X$,

(18_2) $\sigma \cdot y = y$ for $y \in Y$,

(19_1) $x \cdot e = x$ for all $x \in X$,

(19_2) $e \cdot u = u$ for all $u \in U \setminus \{\infty\}$,

(20_1) $x \cdot y = z$ is uniquely solvable in $x \in X \setminus \{\sigma\}$ for given $x \in X \setminus \{\sigma\}$, $z \in Y \setminus \{\sigma\}$.

The condition

(20_2) $x \cdot y = z$ is uniquely solvable in $x \in X \setminus \{\sigma\}$ for given $y \in U \setminus \{\sigma, \infty\}$, $z \in Y$

holds iff (2_2) is satisfied.

Proof. $x + \sigma = x \iff T(x,x,e) = \sigma$ (valid by (13_1)), $\sigma + y = y \iff T(\sigma,x,e) = e$ (valid by (12)), $x + y = z \iff T(x,z,e) = y$ (here, for given y, z a unique solution z exists by (2_1); secondly, for given x, z, the corresponding y is uniquely determined because T is well-defined), $x \cdot \sigma = \sigma \iff T(x,\sigma,\sigma) = \sigma$ (valid by (13_2)), $\sigma \cdot y = y \iff T(\sigma,\sigma,y) = \sigma$ (valid by (12)), $x \cdot e = x \iff T(x,x,e) = \sigma$ (valid by (13_1)), $e \cdot y = y \iff T(e,y,y) = \sigma$ (valid by (13_2)), $x \cdot y = z \iff T(x,z,y) = \sigma$ (here, for given x, z a unique solution y exists by (2_1); similarly for $(20_2) \iff (2_2)$.

Corollary to Propositions 4 and 5 : The condition

(21) $\mathrm{card}\,(A \cap E) = 1$ for all $A \in \mathfrak{X}$ (where E is

defined in the proof of Proposition 4 and $\mathcal{X}$ in (1)) implies $X = Y$. If (21) and (2_2) are satisfied then $U \setminus \{\infty\} = Y$. Thus, in the case that (21) and (2_2) hold, the system $(X, +, \cdot)$ is a double-loop. (5)

Proof. $(21) \Rightarrow \sigma$ is a bijection; $(2_2) \Rightarrow \rho$ is a bijection.

Proposition 6. Let $\mathcal{D} = (X, +, \cdot)$ be a double-loop, and let the map $T: X \times X \times X \cup \{\infty\} \to X$ be defined by the linearity property (6)

(22) $T(a, a \cdot b + c, b) = c$ for all $a, b, c \in X$,

and by $T(a, b, \infty) = a$ for all $a, b \in X$, where ∞ is a new element not belonging to X. If the ternar $\mathcal{T} = (X, X, X \cup \{\infty\}, X, T)$ satisfies (5), then $\mathcal{T}$ is special and satisfies (7_2).

Proof. $\operatorname{card} X \geq 2$ because $\mathcal{D}$ has the zero and unit elements. From the loop properties of $\mathcal{D}$ there follows the remaining conditions (6), (7_1) and (7_2).

Proposition 7. There is a special parallel system $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel)$ of the following type:

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(5) That is, $(X, +)$ is a loop with a neutral element o, $(X \setminus \{o\}, \cdot)$ is a loop with a neutral element e and $x \cdot o = o \cdot x = o$ for all $x \in X$ (cf. [7], p.61).

(6) Compare with [2], p.10 or [4], p.505, respectively.

1) $\mathcal{P}$ satisfies (2_2) but it does not satisfy one of the conditions (3),(4),

2) $\mathcal{P}$ satisfies (3) but not (2_2),

3) $\mathcal{P}$ satisfies (2_2) and (3) but not (4).

<u>Proof</u>. 1) First, note that if $(X,+,\cdot)$ is a neofield (8) with right cancellation, i.e. $a+c=b+c \Rightarrow a=b$, then for every choice of u_1, v_1, u_2, v_2 with $u_1 \neq u_2$ there exists an $x \in X$ such that $x\cdot u_1 + v_1 \neq x \cdot u_2 + v_2$.
Consider the examples of non-planar neofields with right cancellation constructed in [5]. Then, by Proposition 7, we obtain special parallel systems of the required type ((2_2) is valid, one of (3),(4) is not valid).

2) We shall use the examples of $(S,+,\square)$ constructed in [3], and by Proposition 6, obtain a parallel

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(8) $\mathcal{D}=(X,+,\cdot)$ is a <u>neofield</u>, if $(X,+)$ is a loop with neutral element o, $(X\setminus\{o\},\cdot)$ is a group, and both distributivity laws hold (cf.[5],p.40). A neofield $\mathcal{D}$ is called <u>planar</u> if it satisfies the conditions (cf.[5], p. 55):

a) $ax+b=cx+d$ is uniquely solvable in x for given a,b,c,d with $a\neq c$,

b) $x\cdot a+b=x\cdot c+d$ is uniquely solvable in x for given a,b,c,d with $a\neq c$.

system of the required type ((3) valid, (2_2) not valid). One may also apply the procedure of [6], p.337, as follows: A halfcartesian group G_0 may be imbedded in any halfcartesian group G_1 satisfying (5), [6], p.335 for all $a, b, c \in G_0$. G_1 may be imbedded in a halfcartesian group G_2 satisfying (5)[6], p.335 for all $a, b, c \in G_1$. On repeating this process we obtain a sequence $(G_n)_{n=0}^{\infty}$, the union of which is a halfcartesian group $\overline{G}$ satisfying (5)[6], p.335 for all $a, b, c \in \overline{G}$. It may be shown that $\overline{G}$ does not satisfy (4), [6], p.335 if this law is not valid in G_0. And such a G_0 exists: for example, G_0 may be chosen as the ring of integers. From $\overline{G}$, we obtain the desired parallel system using Proposition 6.

3) In this case one may use the last example of [3], and apply Proposition 6 as before.

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