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Coordinatization of parallel systems

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In the present note we shall investigate coordinatizing system to certain types of André's parallel systems.

**Definition 1.** A parallel system is a triplet \( \mathcal{P} = (\mathcal{P}, \mathcal{L}, \parallel) \) where
1. \( \mathcal{P} \) is a nonvoid set of elements called points,
2. \( \mathcal{L} \) is a nonvoid set of some nonvoid subsets in \( \mathcal{P} \) called lines,
3. \( \parallel \) is a partition of \( \mathcal{L} \) such that each \( \mathcal{L} \in \parallel \) is a partition of \( \mathcal{P} \), and
4. there are three points not on the same line.

\( \mathcal{P} \) is called special if \( \mathcal{P} = X \times Y \) for some sets \( X, Y \) and if
(1) \( X = \{(x, y) \mid y \in Y, x \in X \} \in \parallel \), \( Y = \{(x, y) \mid x \in X, y \in Y \} \in \parallel \),
(2) \( \text{card} (A \cap B) = 1 \) for \( A \in \parallel \setminus Y, B \in Y \).

For further application we shall formulate three further conditions:
(2') \( \text{card} (A \cap B) = 1 \) for \( A \in \parallel \setminus X, B \in X \),
(3) if \( A, B \) are distinct points then there is exactly one line containing both \( A, B \),
(4) \( \text{card} (A \cap B) = 1 \) for lines \( A, B \) belonging to distinct elements of \( \parallel \).

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(1) Cf. [1], p. 90.
(2) A partition of a set \( S \neq \emptyset \) is a decomposition of \( S \)
into pairwise disjoint nonvoid subsets in \( S \) covering \( S \).
**Definition 2.** A ternar is a quintuplet $\mathcal{I} = (X, Y, U, V, T)$ where

1) $X, Y, U, V$ are nonvoid sets with $\text{card } X \geq 2$, $\text{card } Y \geq 2$ and 2) $T$ is a map from $X \times Y \times U$ onto $V$.

Set $L(\mu, \nu) = \{ (x, y) \mid T(x, y, \mu) = \nu \}$, $\mathcal{J} = \{ (\mu, \nu) \mid L(\mu, \nu) + \emptyset \neq \emptyset \}$, $\mathcal{L}_\mu$ being the set of all $L(\mu, \nu) + \emptyset$ with fixed $\mu \in U$ and $\| = \{ \mathcal{L}_\mu \mid \mu \in U \}$.

$\mathcal{I}$ is called special if

1. the map $(\mu, \nu) \rightarrow L(\mu, \nu)$ is a bijection of $\mathcal{I}$ onto $\mathcal{L}$,

2. there exist elements $\sigma, \omega \in U$ and injections $\xi: X \rightarrow V, \eta: Y \rightarrow V$ such that $T(x, y, \sigma) = \eta y$ and $T(x, y, \omega) = \xi x$ for all $x \in X, y \in Y$,

3. $T(x, y, \mu) = \nu$ is uniquely solvable in $y \in Y$ for given $x \in X, (\mu, \nu) \in \mathcal{J}$.

Next we formulate some further conditions:

4. $T(x, y, \mu) = \nu$ is uniquely solvable in $x \in X$ for given $y \in Y, (\mu, \nu) \in \mathcal{J}$,

5. if $(x_1, y_1), (x_2, y_2)$ are distinct elements in $X \times Y$, then there is exactly one $\mu \in U$ satisfying $T(x_1, y_1, \mu) = T(x_2, y_2, \mu),$

6. if $(\mu_1, \nu_1), (\mu_2, \nu_2)$ are distinct elements in $\mathcal{J}$, then the equations $T(x, y, \mu_1) = \nu_1, T(x, y, \mu_2) = \nu_2$ have a unique solution $(x, y) \in X \times Y$.

**Proposition 1.** Let $\mathcal{I} = (X, Y, U, V, T)$ be a ternar. Then $I$ (cf. Definition 2) is a partition of $\mathcal{L}$ (cf. Definition 2) iff (5) holds.
Proof. If $L(u_1, v_1) = L(u_2, v_2)$ for some $(u_1, v_1), (u_2, v_2) \in \mathcal{L}$ with $u_1 \neq u_2$ then $(u, v) \mapsto L(u, v)$ is not a bijection of $\mathcal{L}$ onto $\mathcal{L}$. Conversely, if $(u, v) \mapsto L(u, v)$ is a bijection of $\mathcal{L}$ onto $\mathcal{L}$, then $L(u_1, v_1) \neq L(u_2, v_2)$ for distinct $(u_1, v_1), (u_2, v_2) \in \mathcal{L}$.

Proposition 2. Let $\mathcal{T} = (X, Y, U, V, T)$ be a special ternar. Then $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel)$ (cf. Definition 2) is a special parallel system, to be termed associated with $\mathcal{T}$.

Proof. By Proposition 1, $\parallel$ must be a partition of $\mathcal{L}$, and by the definition of $\mathcal{L}_\mu$ (cf. Definition 2), each $\mathcal{L}_\mu, \mu \in U$, is a partition of $X \times Y$. From (6) and (7) there follow (1) and (2). Finally, from $\text{card } X \geq 2$, $\text{card } Y \geq 2$, and by (5), (6), (7), there exist at least four elements of $X \times Y$ which are not on the same line. Thus $\mathcal{P}$ is a special parallel system. Note that for $\mathcal{T}$ and $\mathcal{P}$, $(2_2) \iff (2_2)$, (3) $\iff$ (8) and (4) $\iff$ (9).

Proposition 3. Let there be given a special parallel system $\mathcal{P} = (X \times Y, \mathcal{L}, \parallel), \mathcal{L} = \{\mathcal{L}_\mu \mid \mu \in U\}$. Choose a set $V$ such that there exist injections $\alpha_\mu : \mathcal{L}_\mu \to V$ for all $\mu \in U$ and $V = \bigcup_{\mu \in U} \mathcal{L}_\mu$. Define the map $T$: $X \times Y \times U \to V$ by $T(x, y, \mu) = v \iff (x, y) = \alpha_\mu^\dagger v$. Then the ternar $\mathcal{T} = (X, Y, U, V, T)$ is special (and will be called associated with $\mathcal{P}$).

Proof. As $\parallel$ is a partition of $\mathcal{L}$, (5) holds by
Proposition 1. Furthermore, \( (1) \Rightarrow (\delta) \) and \( (2_\gamma) \Rightarrow (\gamma) \).

From the fact that there are at least three points which are not on the same line, it follows that \( \text{card } X \geq 2 \), \( \text{card } Y \geq 2 \). Thus \( \mathcal{F} \) is a special ternar. Note that, for \( \mathcal{P} \) and \( \mathcal{F} \), \( (2_2) \Rightarrow (\gamma) \), \( (3) \Rightarrow (\delta) \) and \( (4) \iff (3) \).

**Proposition 4.** Let \( \mathcal{P} = (X \times Y, R, \Pi) \), \( R = \{ R_U \}_{U \in \mathcal{V}} \) be a special parallel system. Then an associated special ternar \( \mathcal{F} = (X, Y, U, V, T) \) can be chosen such that (using the notation of Definition 2)

\[
\begin{align*}
(10) & \quad \sigma \in X \leq Y = V \supseteq U \setminus \{ \infty \}, \\
(11) & \quad T(x, y, \sigma) = y, T(x, y, \infty) = x \quad \text{for all } x \in X, y \in Y, \\
(12) & \quad T(\sigma, v, \mu) = v \quad \text{for all } \mu \in U, v \in V, \\
(13) & \quad \text{there is an element } e \in X \setminus \{ \sigma \} \text{ satisfying} \\
& \quad T(x, x, e) = \sigma, T(e, \mu, \mu) = e \quad \text{for all } x \in X, \mu \in U \setminus \{ \infty \}.
\end{align*}
\]

**Proof.**

a) Choose a point \( O = (\alpha_1, \alpha_2) \) and a line \((x, y)\) with \( e \neq \alpha \). Let \( \mathcal{P} \) be the injection of \( U \setminus \{ \infty \} \) into \( Y \) defined as follows: For each \( \mu \in U \setminus \{ \infty \} \), let \( (e, \mu) \in \mathcal{L} \), where \( 0 \in \mathcal{L} \subseteq \mathcal{L}_u \). Thus we can identify each \( \mu \in U \setminus \{ \infty \} \) with \( g \mu \), and obtain \( U \setminus \{ \infty \} \equiv Y \).

b) Choose a line \( E \) with \( 0 \in E \in \mathcal{L}_{\mathcal{E}_2} \) for some \( e_2 \in \mathcal{E}_{\{ \alpha, \infty \}} \), and define an injection \( \sigma: X \rightarrow Y \) by \( (x, \sigma x) \in E \) for each \( x \in X \). Then we can identify each \( x \in X \) with \( \sigma x \), and obtain \( X \equiv Y \). After this identification, we have \( \sigma = \alpha_1 = \alpha_2, e = e_2 \).

c) It is possible to take \( \xi, \eta \) as identity maps.
Then (11) is satisfied.

d) Let each line \( L \in \mathcal{L}_u \), \( u \in U \setminus \{ \infty \} \), be uniquely determined by the "intercept" \((\sigma, \nu) \in L\) so that
\[ L = \{(x,y) \mid T(x,y,u) = \alpha_u L \} \cdot \]
We have the bijection \( \lambda_u : V \rightarrow Y \) where \( \lambda_u (\alpha_u L) = \nu \) for each \( L \in \mathcal{L}_u \). After identifying each \( \alpha_u L \) with \( \lambda_u (\alpha_u L) \), we obtain \( V = Y \). Thus (10) is proved.

e) \((\sigma, \nu) \in \{(x,y) \mid T(x,y,u) = \nu \} \Rightarrow (12), \]
\[ E = \{(x,y) \mid T(x,y,e) = \sigma \} \Rightarrow (13) \]
and \((e,y) \in \{(x,y) \mid T(x,y,e) = \sigma \} \Rightarrow (13)\).

**Proposition 5.** Let \( \mathcal{P} = (X \times Y, \mathcal{L}, \parallel) \) be a special parallel system, and let \( \mathcal{T} = (X, Y, U, Y, T) \) be the associated special ternary constructed in Proposition 4.

Define two derived maps \( X \times Y \rightarrow Y \) (denoted as addition) and \( X \times U \rightarrow Y \) (denoted as multiplication) by

\[
(14) \quad T(x, x + y, e) = y \quad \text{for } x \in X, \ y \in Y, \\
(15) \quad T(x, x \cdot y) = \sigma \quad \text{for } x \in X, \ y \in U.
\]

Then

\[
(16_1) \quad x + \sigma = x \quad \text{for } x \in X, \\
(16_2) \quad \sigma + y = y \quad \text{for } y \in Y, \\
(17_1) \quad x + y = x \quad \text{is uniquely solvable in } x \in X \quad \text{for given } y \in Y, \ x \in Y,
\]

3) Compare with [2], p. 5 or [4], p. 503, respectively.

4) Compare with [4], p. 505.
(17) \( X + y = z \) is uniquely solvable in \( y \in Y \)
for given \( x \in X, \ z \in Y \),
(18) \( x \cdot e = x \) for \( x \in X \),
(19) \( \sigma \cdot y = y \) for \( y \in Y \),
(20) \( x \cdot e = x \) for all \( x \in X \),
(21) \( e \cdot u = u \) for all \( u \in U \setminus \{\infty\} \),
(22) \( x \cdot y = x \) is uniquely solvable in \( x \in X \setminus \{\sigma\} \)
for given \( x \in X \setminus \{\sigma\}, \ y \in Y \setminus \{\sigma\} \).

The condition

(22) \( x \cdot y = x \) is uniquely solvable in \( x \in X \setminus \{\sigma\} \)
for given \( y \in U \setminus \{\sigma, \infty\}, \ x \in Y \)
holds iff (22) is satisfied.

Proof. \( x + \sigma = x \Longleftrightarrow T(x, x, e) = \sigma \) (valid by (13)),
\( \sigma + y = y \Longleftrightarrow T(\sigma, x, e) = e \) (valid by (12)), \( x + y = x \Longleftrightarrow \)
\( \Longleftrightarrow T(x, x, e) = y \) (here, for given \( y, x \) a unique solution \( x \) exists by (21); secondly, for given \( x, y \), the
the corresponding \( y \) is uniquely determined because \( T \) is well-
defined), \( x \cdot \sigma = \sigma \Longleftrightarrow T(x, \sigma, \sigma) = \sigma \) (valid by
(13)), \( \sigma \cdot y = y \Longleftrightarrow T(\sigma, \sigma, y) = \sigma \) (valid by (12)),
\( x \cdot e = x \Longleftrightarrow T(x, x, e) = \sigma \) (valid by (13)), \( e \cdot y = y \Longleftrightarrow 
\Longleftrightarrow T(e, y, y) = \sigma \) (valid by (13)), \( x \cdot y = x \Longleftrightarrow T(x, x, y) = \sigma \)
(here, for given \( x, x \) a unique solution \( y \) exists by
(21)); similarly for (22) \( \Longleftrightarrow (22) \).

Corollary to Propositions 4 and 5: The condition
(21) \( \text{card} (A \cap E) = 1 \) for all \( A \in X \) (where \( E \) is
defined in the proof of Proposition 4 and \( \mathcal{X} \) in \((1)\) implies \( X = Y \). If \((21)\) and \((22)\) are satisfied then \( U \triangleright \{\omega\} = Y \). Thus, in the case that \((21)\) and \((22)\) hold, the system \((X, +, \cdot)\) is a double-loop. \((5)\)

**Proof.** \((21) \Rightarrow \sigma \) is a bijection; \((22) \Rightarrow \rho \) is a bijection.

**Proposition 6.** Let \( \mathcal{D} = (X, +, \cdot) \) be a double-loop, and let the map \( T : X \times X \times X \cup \{\omega\} \to X \) be defined by the linearity property \((6)\)

\[
T(a, a \cdot b + c, b) = c \quad \text{for all } a, b, c \in X,
\]
and by \( T(a, t, \omega) = a \) for all \( a, \omega \in X \), where \( \omega \) is a new element not belonging to \( X \). If the ternary \( \mathcal{T} = (X, X, X \cup \{\omega\}, X, T) \) satisfies \((5)\), then \( \mathcal{T} \) is special and satisfies \((7_2)\).

**Proof.** \( \text{card } X = 2 \) because \( \mathcal{D} \) has the zero and unit elements. From the loop properties of \( \mathcal{D} \) there follows the remaining conditions \((6),(7_1)\) and \((7_2)\).

**Proposition 7.** There is a special parallel system \( \mathcal{P} = (X \times \mathcal{L}, \parallel)\) of the following type:

\[
(5) \text{ That is, } (X, +) \text{ is a loop with a neutral element } \sigma, \\
( X \setminus \{\sigma\}, \cdot) \text{ is a loop with a neutral element } e \text{ and } x \cdot \sigma = \sigma \cdot x = \sigma \text{ for all } x \in X \text{ (cf.}[7], p.61). \\
(6) \text{ Compare with [2], p.10 or [4], p.505, respectively.}
\]
1) $\mathcal{P}$ satisfies $(2)$ but it does not satisfy one of the conditions $(3),(4)$,
2) $\mathcal{P}$ satisfies $(3)$ but not $(2)$,
3) $\mathcal{P}$ satisfies $(2)$ and $(3)$ but not $(4)$.

Proof. 1) First, note that if $(X,+)$ is a neo-field $(8)$ with right cancellation, i.e. $a+c=b+c \Rightarrow a=b$, then for every choice of $\mu_1, \nu_1, \mu_2, \nu_2$ with $\mu_1+\mu_2$ there exists an $x \in X$ such that $x \cdot \mu_1 \mu_2 + \nu_1 \mu_2 + \nu_2$.
Consider the examples of non-planar neofields with right cancellation constructed in [5]. Then, by Proposition 7, we obtain special parallel systems of the required type ($(2)$ is valid, one of $(3),(4)$ is not valid).

2) We shall use the examples of $(S,+,\cdot)$ constructed in [3], and by Proposition 6, obtain a parallel

(8) $\mathcal{D}=(X,+)$ is a neo-field, if $(X,+)$ is a loop with neutral element $\sigma$, $(X \setminus \{\sigma\})$ is a group, and both distributivity laws hold (cf.[5],p.40). A neofield $\mathcal{D}$ is called planar if it satisfies the conditions (cf.[5], p. 55):
a) $a \cdot x + b = c \cdot x + d$ is uniquely solvable in $X$ for given $a,b,c,d$ with $a \neq c$,
b) $x \cdot a + b = x \cdot c + d$ is uniquely solvable in $X$ for given $a,b,c,d$ with $a \neq c$.
system of the required type ((3) valid, (2) not valid).
One may also apply the procedure of [6], p.337, as follows: A halfcartesian group $G_0$ may be imbedded in any halfcartesian group $G_1$ satisfying (5), [6], p.335 for all $a, b, c \in G_1$. $G_1$ may be imbedded in a halfcartesian group $G_2$ satisfying (5) [6], p.335 for all $a, b, c \in G_1$. On repeating this process we obtain a sequence $(G_m)_{m=0}^{\infty}$, the union of which is a halfcartesian group $G$ satisfying (5) [6], p.335 for all $a, b, c \in G$. It may be shown that $G$ does not satisfy (4), [6], p.335 if this law is not valid in $G_0$. And such a $G_0$ exists: for example, $G_0$ may be chosen as the ring of integers. From $G$, we obtain the desired parallel system using Proposition 6.

3) In this case one may use the last example of [3], and apply Proposition 6 as before.

References


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