Zdeněk Hedrlín; Petr Vopěnka

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AN UNDECIDABLE THEOREM CONCERNING FULL EMBEDDINGS INTO CATEGORIES OF ALGEBRAS

Z. HEDRLÍN and P. VOPĚNKA, Praha

Similarly as in [3], a category which is isomorphic with a full subcategory of algebras is called boundable. In [4] J.R. Isbell raised a question to find a concrete category which is not boundable. The aim of the present note is to show that the boundability of a category depends on the used set theory. The category, given as an example, is the category of sets with inclusions. It is not boundable in a (rather odd) set theory and boundable in a usual one, in which the last result implies e.g. the following theorem: to any set A there exists a grupoid (graph, topological space, resp.) $G(A)$ such that $A \subset B$ is equivalent with the existence of exactly one homomorphism (graph-homomorphism, local homeomorphism, resp.) from $G(A)$ into $G(B)$ and if $A \not\subset B$, then it does not exist.

In any set theory, the inclusions as morphisms and sets as objects form a category, which we designate by $\mathcal{S}_i$. By a concrete category we mean any category, which is isomorphic with a subcategory of sets and their mappings $\mathcal{S}$. Evidently, $\mathcal{S}_i$ is a concrete category. It turns out that the boundability of $\mathcal{S}_i$ depends substantially on the following axiom:

(V) There is one-to-one mapping F of the universal class V onto the class of all ordinals On.

We shall work in the set theory Σ_0^*, i.e. in the Gödel-Bernays set theory with the axioms of groups A, B, C and the axiom of choice E. We shall need also the following axiom:

(M) There is a cardinal δ such that two valued δ-additive measure on any set is γ-additive for any cardinal γ.

The main result of this note may be described by two theorems:

__Theorem 1.__- In the set theory $\Sigma_0^* + (V) + (M)$, $\mathcal{T}_i$ is boundable.

__Theorem 2.__ In the set theory $\Sigma_0^* + (non\ V)$, $\mathcal{T}_i$ is not boundable.

It is easy to see that, if Σ_0^* is consistent, $\Sigma_0^* + (M) + (V)$ is consistent. Really, it follows from the consistency of Σ_0^* that Σ^* (Σ^* denotes Σ_0^* + the axiom of regularity D) is consistent. The axiom (V) is provable in Σ^*. Denote by (I) the following axiom:

(I) There exists an inaccessible cardinal.

Then, if Σ^* is consistent, $\Sigma^* + (M)$ is consistent, as e.g. Σ^* consistent implies $\Sigma^* + (non\ I)$ is consistent, and in the last theory (M) is provable even for $\delta = \aleph_0$.

If $\Sigma_0^* + (I)$ is consistent, then $\Sigma_0^* + (non\ V)$ is consistent. Really, in the set theory $\Sigma_0^* + (I)$ it is possible to construct a model of $\Sigma_0^* + (non\ V)$.

Thus, we may derive the following corollary:

<u>Corollary</u>. If (I) is undecidable in Σ_0^*, then the assertion " $\mathcal{T}_i$ is boundable" is undecidable in Σ_0^*.

To prove theorem 1, we use a result of [3] and the construction defined in [1]. The idea of the proof of theorem 2 is very simple.

<u>Proof of theorem 1</u>.

Denote by $\mathcal{C}$ the following category: the objects are all non-limit ordinals, α, $\alpha > 1$. On every object - we remark that, by definition, an ordinal α is the set of all ordinals, β, $\beta < \alpha$ - there is exactly one morphism, namely the identity transformation of α, and there are no other morphisms in $\mathcal{C}$.

<u>Lemma 1</u>. Assuming (M), $\mathcal{C}$ is a boundable category. Hence, $\mathcal{C}$ is isomorphic with a full subcategory of $\mathcal{R}$ (for definition see [2] or [3]).

<u>Proof</u>. We shall show that $\mathcal{C}$ is a full subcategory of $\mathcal{T}((P^-,\{2\}), (I,\{1\}))$ defined in [3].

Really, in [3] it has been proved that the category $\mathcal{W}$ - the trivial category of ordinals - is a full subcategory of $\mathcal{T}(P^-,\{2\})$ by introduction of a binary relation r on $P^-(\alpha)$, α an arbitrary ordinal. If α is a non-limit ordinal, define on $P^-(\alpha)$ the binary relation r, and a unary relation on α "to be the greatest element of α ". If α, β are non-limit ordinals, $f: \alpha \rightarrow \beta$, then $P^-(f)$ is compatible with the relations r if and only if $\alpha \leq \beta$ and f is a natural inclusion. Now, $f : \alpha \to \beta$ is a morphism in $\mathcal{T}((P^-,\{2\}),(I,\{1\}))$, if and only if $\alpha \leq \beta$, f is a natural inclusion and the last element in α is sent by f into the last element of β. Hence, $\alpha = \beta$. By [3], $\mathcal{T}((P^-,\{2\})(I,\{1\}))$ is boundable, and by [2], $\mathcal{V}$ can be fully embedded into $\mathcal{R}$.

Definition of disjoint sum of sets and relations. Let K be a class of ordinals. For every $\alpha \in K$, let X_α be a set, R_α a binary relation on X_α. If A is a set, $A \subset K$, we define a set $\underset{\alpha \in A}{D} X_\alpha$ (a disjoint union) by:

$$\underset{\alpha \in A}{D} X_\alpha = \{(x,\alpha) \mid \alpha \in A,\ x \in X_\alpha\},$$

and a binary relation $\underset{\alpha \in A}{D} R_\alpha$ on $\underset{\alpha \in A}{D} X_\alpha$ by:

$$((x,\alpha),(y,\beta)) \in \underset{\alpha \in A}{D} R_\alpha \Leftrightarrow \alpha = \beta,\ (x,y) \in R_\alpha .$$

We designate by K_0 the class of all non-limit ordinals α, $\alpha > 1$.

Lemma 2. There exists a class of couples (X_α, R_α), X_α a set, R_α a binary relation on X_α, $\alpha \in K_0$, with the following property:

if A, $B \subset K_0$ are sets, $f : \underset{\alpha \in A}{D} X_\alpha \to \underset{\beta \in B}{D} X_\beta$ such that

(1) $((x,\alpha),(y,\alpha')) \in \underset{\alpha \in A}{D} R_\alpha \Rightarrow (f((x,\alpha)), f((y,\alpha'))) \in \underset{\beta \in B}{D} R_\beta$,

then $A \subset B$ and $f((x,\alpha)) = (x,\alpha)$ for every $(x,\alpha) \in \underset{\alpha \in A}{D} X_\alpha$.

Proof. By lemma 1, $\mathcal{V}$ is isomorphic with a full subcategory of $\mathcal{R}$. It means that, with every $\alpha \in K_0$, we may associate a set Y_α and a binary relation S_α on Y_α such that if $\alpha, \beta \in K_0$, $f : Y_\alpha \to Y_\beta$ which is

$S_\alpha S_\beta$ -compatible, then $\alpha = \beta$ and f is the identity transformation. We remark that these sets and relations need not fulfil the condition of lemma 2.

If R is a binary relation on a set X, we define a relation $\hat{R}$ on X by: $(x, x') \in \hat{R}$ if and only if one of the following conditions holds: $x = x'$, $(x, x') \in R$, $(x', x) \in R$. A couple (x, x') is called to belong to the same component according to $\hat{R}$ - we write $(x, x') \in \in C(\hat{R})$ - if there exists a finite sequence $x_1, x_2, \dots, x_n$ such that $x = x_1$, $x' = x_n$, $(x_i, x_{i+1}) \in \hat{R}$ for $i = 1, 2, \dots, n-1$. The relation $C(\hat{R})$ is an equivalence relation and their equivalence classes are called components of R. Evidently, every compatible mapping sends each component into a component. A relation R on X is called connected, if there is only one equivalence class according to $C(\hat{R})$, namely X.

Observe, that if all relations S_α on Y_α, $\alpha \in K_0$, are connected, then they fulfil the condition of lemma 2. Really, by definition of $\underset{\alpha \in A}{D} S_\alpha$, (x, α) and (y, β) cannot be in the relation $C(\underset{\alpha \in A}{\hat{D}} S_\alpha)$ for $\alpha \neq \beta$. Hence, the components according to $\underset{\alpha \in A}{D} S_\alpha$ are exactly the sets $\{(x, \alpha) \mid \alpha$ fixed, x arbitrary in $Y_\alpha\}$. By definition, the relation $\underset{\alpha \in A}{D} S_\alpha$ restricted to the component of $\underset{\alpha \in A}{D} S_\alpha$ defined by α is isomorphic with the relation S_α on Y_α. Now, let f fulfil the implication (1) of the lemma. Then f must map every component of $\underset{\alpha \in A}{D} Y_\alpha$ according to $\underset{\alpha \in A}{D} S_\alpha$ into a component of $\underset{\beta \in B}{D} Y_\beta$ according to

$\underset{\beta \in B}{D} S_\beta$. As the components are isomorphic with (Y_α, S_α), we get that, for every $\alpha \in A$, there is $\beta \in B$ such that the restriction of $\underset{\alpha \in A}{D} S_\alpha$ onto a component defined by α - say f_α - is a mapping from Y_α into Y_β which is $S_\alpha S_\beta$ -compatible. But it is possible if and only if $\alpha = \beta$ and f_α is the identity. Hence, lemma 2 would be proved, if all the relations S_α on Y_α are connected. But, generally, the relations need not be connected. It is the reason, why we use the construction from [1], which will change all the relations into connected ones, leaving them all the useful properties.

If S_α is a relation on a set Y_α , we define a set X_α and a relation R_α on X_α by the construction in [1] putting $X = Y_\alpha$, $i = 1$, $R_1 = S_\alpha$ (this is the reason why we have assumed $\alpha \in K_o$ implies $\alpha > 1$), $t_1 = 2$, $X_T = X_\alpha$, $R_T = R_\alpha$. Using the same method as in [1] it is easy to prove that $f : X_\alpha \to X_\beta$, $\alpha, \beta \in K_o$, is a $R_\alpha R_\beta$-compatible mapping if and only if $\alpha = \beta$ and f is the identity. Moreover, by definition, it is evident that every R_α, $\alpha \in K_o$, is a connected relation. Hence, the relations R_α on X_α fulfil the requirements of lemma 2.

Now, we can complete the proof of theorem 1.

Let F be a one-to-one mapping of the universal class V into the class of all non-limit ordinals K_o, $\alpha \in K_o$ implies $\alpha > 1$. Hence, for any set X , we get an ordinal $\alpha = F(X)$, $\alpha \in K_o$. Put $G(X) = X_{F(X)}$, $H(X) = R_{F(X)}$, where X_α and R_α have the properties from lemma 2. Now, if Y is a set, put

$$S(Y) = \underset{X \in Y, X = \emptyset}{D} G(X), \qquad R(Y) = \underset{X \in Y, X = \emptyset}{D} H(X)$$

(if we consider only non-void sets, the void set in the union may be omitted). It follows from lemma 2, that, if Y_1 and Y_2 are sets, then there exists a $R(Y_1)R(Y_2)$-compatible mapping from $S(Y_1)$ into $S(Y_2)$ if and only if $Y_1 \subset Y_2$, which is then the natural inclusion of $S(Y_1)$ into $S(Y_2)$. We have constructed a full embedding of $\mathcal{N}_i$ into $\mathcal{R}$. It has been proved in [2], that $\mathcal{R}$ can be fully embedded into the category of algebras with e.g. two unary operations. The proof of theorem 1 is completed.

<u>Proof of theorem 2</u>. First, we shall prove a lemma.

<u>Lemma 3</u>. In the set theory Σ_0^*, any class of mutually non-isomorphic algebras of an arbitrary fixed type can be mapped by a one-to-one mapping into the class of all ordinals O_n.

<u>Proof</u>. Let Δ be the type of the algebras. As any algebra is isomorphic with an algebra defined on a cardinal α, we may consider only algebras defined on cardinals. If α is a cardinal, then there is only a set $M(\alpha)$ of algebras of the type Δ defined on α. If $\alpha \neq \beta$, then $M(\alpha) \cap M(\beta) = \emptyset$. Denote by S_α the set of all well orderings of M_α, $S = \cup S_\alpha$, where the sum is taken over cardinals. By the axiom of choice, there exists a function AS associating with every cardinal α

a well ordering $K(\alpha)$ of the set $M(\alpha)$. Put $M = \bigcup M_\alpha$. If $y \in M$ there is exactly one cardinal α such that $y \in$ $\in M(\alpha)$. We designate this cardinal by $y(\alpha)$. Now, define on M a lexicographical ordering $\prec$ by:

$x \prec y$, $x, y \in M$ if and only if $x(\alpha) < y(\alpha)$ or $x(\alpha) = y(\alpha) = \alpha$ and $x < y$ in the ordering $K(\alpha)$.

The ordering $\prec$ is evidently a well ordering, and, for every x, the class of all y, $y \prec x$, is a set. Hence, the class M can be mapped by a one-to-one mapping into $\mathcal{O}_n$. The lemma follows.

Lemma 3 enables us to conclude the proof of the theorem 2. Consider the class of all one-element sets Z. There is one-to-one mapping G of the universal class V onto Z, namely $G(X) = \{X\}$, for every $X \in V$. By assumption, there is no one-to-one mapping of V into $\mathcal{O}_n$.

If $\mathcal{T}_i$ is boundable, then for any one point set $\{X\}$ we get an algebra $A(X)$ of a fixed type Δ. If X, Y are sets, $X \neq Y$, then $\{X\} \not\subset \{Y\}$ and $\{Y\} \not\subset \{X\}$. Therefore $A(X)$ and $A(Y)$ must not be isomorphic. By lemma 3, any class of non-isomorphic algebras of a given type may be mapped by one-to-one mapping into $\mathcal{O}_n$. Hence, Z can be mapped by a one-to-one mapping into $\mathcal{O}_n$ – a contradiction. The proof of theorem 2 is finished.

<u>Remark</u>. If $\mathcal{Q}$ and $\mathcal{L}$ are subcategories of the category of sets and mappings $\mathcal{T}$, $F : \mathcal{Q} \to \mathcal{L}$ a functor which maps $\mathcal{Q}$ onto a full subcategory of $\mathcal{L}$, F is called limited, if for every cardinal α, there is a cardinal β such that $card\ X = \alpha$ implies $card\ F(X) \leq \beta$.

Evidently, $\mathcal{T}_i$ may be considered as a subcategory of $\mathcal{T}$. On the other hand, it is easy to see that $\mathcal{T}_i$ cannot be fully embedded into $\mathcal{R}$ by a limited functor. Thus, the functor which maps $\mathcal{T}_i$ onto a full subcategory of $\mathcal{R}$ in the set theory $\Sigma_o^* + (V) + (M)$ is an example of a functor which is not limited.

R e f e r e n c e s

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