Consistency theorems connected with some combinatorial problems

Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 4, 495--499

Persistent URL: http://dml.cz/dmlcz/105082

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1966

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
CONSISTENCY THEOREMS CONNECTED WITH SOME COMBINATORIAL PROBLEMS

Lev BUKOVSKÝ, Košice

The main purpose of this note is to prove the consistency of the positive solution of a problem of G. Kurepa. The terminology and notation are those of [2], [3]. For notions from partition calculus see [1].

We say that the set $X$ possesses property $(K, \alpha)$ iff

1. $X \in \mathcal{P}(\omega_\alpha)$,
2. $X \in \mathcal{P}^{\omega_{\alpha+1}}$,
3. $(\forall \psi)(\psi \in \omega_\alpha \& \psi < \omega_\alpha \Rightarrow \{x \cap \psi : x \in X \} < \omega_\alpha)$.

G. Kurepa has stated the following problem:

Is there a set $X$ with property $(K,1)$?

The positive solution of this problem leads to many other theorems (for example $\omega_\alpha \rightarrow [\omega_\alpha]_\alpha \rightarrow [\omega_\alpha]_\omega$, see [1], p.154). If $\omega_\alpha$ is strongly inaccessible, then every set with properties $(1, \alpha)$ and $(2, \alpha)$ also possesses property $(3, \alpha)$.

Theorem. Suppose that

1. $\omega_\alpha$ is an inaccessible cardinal in the sense of Gödel's $\Delta$-model,
2. in the $\Delta$-model, there is no cardinal between $\omega_\alpha$ and $\omega_{\alpha+1}$,
3. $\omega_\alpha$ is regular.

Then the set $X = \mathcal{P}(\omega_\alpha) \cap L$ (i.e. the set of all con-
structible subsets of $\mathcal{C}$ possesses the property $(K, \kappa)$.

**Proof.** From (5), $X \sim \mathcal{C}_4 + 1$. Let $\gamma = \alpha_\kappa \cup \beta < \kappa \kappa$. Since $\alpha_\kappa$ is regular, then there is a $\beta \in \alpha_\kappa$ such that $\gamma \leq \beta$. Using (4), we may suppose that $\beta$ is a cardinal number in the sense of the $\Delta$-model. We have to prove that $Y = \{x \cap \gamma, x \in X\}$ is of power less than $\kappa_\kappa$. Set $f(x \cap \gamma) = \beta \cap x$ for $x \in X$. Thus $f$ is a one-to-one mapping of $Y$ into $\mathcal{P}(\beta) \cap L$ (where $L$ is the class of all constructible sets). Let $\gamma$ be the first cardinal number greater than $\beta$ in the sense of the $\Delta$-model. Then there is a one-to-one mapping of $\mathcal{P}(\beta) \cap L$ onto $\gamma$. Hence there is a one-to-one mapping of $Y$ into $\gamma$. Since $\gamma \in \alpha_\kappa$ (using (4)), $\gamma < \kappa_\kappa$. This completes the proof.

Conditions (4) and (5) hold in the model $\mathcal{N}$ constructed in [4] (with $\alpha = \Lambda$, see p. 441). Thus, we have the following

**Metatheorem.** Let $\Lambda$ be a particular ordinal number (in the sense of [3]) such that the regularity of $\alpha_\Lambda$ is provable in the set theory $\Sigma^*$. If the theory $\Sigma^*$ with the axiom "there is an inaccessible cardinal greater than $\alpha_\Lambda$" is consistent, then the theory $\Sigma^*$ with the axiom "there is a set with property $(K, \Lambda + 1)$" is also consistent.

**Corollary.** If the existence of an inaccessible cardinal greater than $\alpha_\Lambda$ is consistent with $\Sigma^*$, then in $\Sigma^*$ it cannot be proved that

$$\kappa_{\Lambda + 2} \rightarrow [\kappa_{\Lambda + 1}]^2_{\kappa_{\Lambda + 1}, \kappa_{\Lambda}}$$

**Proof.** It suffices to prove that the existence of a set $X$ with property $(K, \alpha + 1)$ implies $\kappa_{\alpha + 2} \rightarrow [\kappa_{\alpha + 1}]^2_{\kappa_{\alpha + 1}, \kappa_{\alpha}}$. 

- 496 -
This is well known. I shall sketch the proof suggested to me by Mr. Hajnal.

By definition, $\kappa_{\alpha+2} \rightarrow [\kappa_{\alpha+1}]^2 \kappa_{\alpha+1}$, $\kappa_\alpha$ is equivalent to the following sentence:

There is a partition $J_\alpha$, $\forall \alpha \in \kappa_{\alpha+1}$ of $[X]^2$, $X = \kappa_{\alpha+2}$ such that for every $A \subseteq X$, $D \subseteq \kappa_{\alpha+1}$, if $\bar{A} = \kappa_{\alpha+1}$, $\bar{D} \subseteq \kappa_\alpha$, then $[A]^2 \not\subseteq \cup_{\gamma \in D} J_\gamma$ (see [1], p.144).

Now, we define such a partition. Let $X$ possess the property $(\kappa, \alpha+1)$. Set

$\{x, y \in J_\alpha \mid x \subseteq \kappa_{\alpha+1}, (x \cdot y) \cup (y \cdot x) \in \gamma \}$ for $\gamma \in \kappa_{\alpha+1}$

Since $x \in X \rightarrow x \subseteq \kappa_{\alpha+1}$, one has $\bigcup_{\gamma \in \kappa_{\alpha+1}} J_\gamma = [X]^2$.

Suppose that there are $A \subseteq X$, $D \subseteq \kappa_{\alpha+1}$, $\bar{A} = \kappa_{\alpha+1}$, $\bar{D} \subseteq \kappa_\alpha$ such that $[A]^2 \subseteq \bigcup_{\gamma \in D} J_\gamma$. Thus, if $x, y \in A$, then $((x \cdot y) \cup (y \cdot x)) \cap D \neq \emptyset$. Set $\gamma = \{x \cap D : x \in A\}$. If $x, y \in A$, then $x \cap D = y \cap D$, therefore $\bar{\gamma} = \kappa_{\alpha+1}$ - a contradiction with $(3, \alpha+1)$.

Consistency of many other assertions may be proved, for example the following

**Metatheorem.** If the existence of an inaccessible cardinal is consistent with $\Sigma^*$, then $\Sigma^*$ with the axiom $\kappa_3 \rightarrow [\kappa_1]^2 \kappa_2$, $\kappa_\alpha$ (and $2^{\kappa_0} = \kappa_2$, $2^{\kappa_1} = \kappa_3$) is consistent.

**Proof.** From [4],[6] it follows that there is a model of the theory $\Sigma^*$ in which: $2^{\kappa_0} = \kappa_2$, $2^{\kappa_1} = \kappa_3$, $\omega_\alpha$ is an inaccessible cardinal in the sense of the $\Delta$-model, there are no cardinals in the sense of the $\Delta$-model between $\omega_\alpha$, $\omega_2$ and between $\omega_2$, $\omega_3$, there is a perfect class $M$. 

- 497 -
(i.e. M is almost universal, complete and closed with respect to the fundamental operations, see [3], p.324) such that \( \mathcal{P}(\omega_1) \cap M = \mathcal{N}_3 \), \( \omega_1 \) is (strongly) inaccessible in the sense of \( M \).

To prove the metatheorem, it suffices to define a partition of \( [\mathcal{P}(\omega_1) \cap M]^2 \):

\[
\mathcal{J}_x = \{ \{ y, z \} : y, z \in \mathcal{P}(\omega_1) \cap M \land (y - x) \cup (x - y) \cap x = 0 \} \text{ for } x \in \omega_1, x = x.
\]

The connection between Kurepa's problem and Mycielski's axiom of determinateness (see [5]) may be interesting, because Mycielski's axiom (A) implies (4) for \( \alpha = 1 \).

Some generalizations of results of this paper will be published later.

I should like to express my thanks to Mr. Hajnal for valuable advice.

References


[4] K. HRBAČEK: Model \( \mathcal{V} [\omega_\lambda \rightarrow \omega_\beta] \) in which \( \beta \) is limit number, CMUC, 6(1965),439-442.


(Received September 12, 1966)