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METRIC SPACES OVER PARTIALLY ORDERED SEMI-GROUPS

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1. Introduction. All axioms for an ordinary metric space can be meaningfully formulated for an abstract metric space, where the abstract metric takes on values in a partially ordered semi-group of a certain type to be defined below. Such a space will be called an MPS-space (see definition below). In general, this abstract metric does not enable one to define a topology, but it does enable one to define convergence. The purpose of this paper is to show that every T_1-space is an MPS-space. We shall also give an example of an MPS-space which is not a topological space. Thus, the class of MPS-spaces includes most topological spaces as well as "spaces" which are not topological spaces. Our work actually generalizes the recent work of Antonovskii, Boltyanskii, and Sarymsakov [1], who have shown that every Hausdorff uniform space can be "metrized" over a semi-field (this is their terminology for an algebraic object much like a partially ordered topological ring). It is easy to translate their results into our terminology: every Hausdorff uniform space is an MPS-space. Finally, in the last section we formulate a general metrization problem.

2. Basic definitions. In this paper we consider only commutative semi-groups S with identity. The binary ope-

ration in S will be denoted by $+$ and the identity by 0. The partial ordering $\leqq$ in S is always assumed to satisfy the following conditions:

a) $0 \leqq \alpha$ for all $\alpha \in S$;

b) $\alpha \leqq \beta$ implies $\alpha + \gamma \leqq \beta + \gamma$ for all $\alpha, \beta, \gamma \in S$;

c) if M is a non-empty subset of S , then $\inf M$ exists.

It follows from c) that if M is a non-empty bounded subset of S , then $\sup M$ exists.

It is clear that the set of non-negative real numbers with the usual addition and ordering is an example of a partially ordered semi-group as defined above. Other examples will be given as they are needed.

A subset M of S is said to be directed to 0 if for $\alpha, \beta \in M$ there exists $\gamma \in M$ such that $\gamma \leqq \alpha$ and $\gamma \leqq \beta$, and if $\inf M = 0$. Addition in S is said to be continuous at 0 if when M and N are directed to 0 , so is $M + N$. It will be seen below that continuity of addition at 0 plays an important role in the study of MPS-spaces.

<u>Definition of an MPS-space</u>. Let X be a non-empty set and S a partially ordered semi-group. Let $d: X \times X \rightarrow S$ be such that for all $x, y, z \in X$

1) $d(x, y) = d(y, x)$;

2) $d(x, z) \leqq d(x, y) + d(y, z)$;

3) $d(x, y) = 0$ iff $x = y$.

A net $\{x_n, n \in D\}$ of elements from X is said to converge to $x \in X$ iff $\lim \sup d(x, x_n) = 0$. The set X with convergence defined in this way will be called an MPS-space. The function $d(\cdot\,,\cdot)$ will be called a PS-metric.

Remark. In writing $\lim\sup d(x, x_n) = 0$, we only assume there exists $k \in D$ such that the set $\{d(x, x_n): k < n \in D\}$ is bounded. We do not assume that the set $\{d(x, x_n): n \in D\}$ is bounded.

3. Examples. It is clear that an ordinary metric space is an MPS-space. An example of an MPS-space which is not a topological space is the following: let X be the collection of all real-valued measurable functions on $[0,1]$ and let S be the partially ordered semi-group of non-negative bounded real-valued measurable functions on $[0,1]$ and let the PS-metric $d(\cdot\,,\cdot)$ be defined as follows:

$$d(x,y) = \frac{|x-y|}{1+|x-y|} \in S \quad \text{for each } x, y \in X .$$

In this way X becomes an MPS-space in which convergence (with respect to d) is just convergence almost everywhere for sequences. It is well known that there is no topology for this convergence.

4. Some general theorems. We state without proof the following theorem.

Theorem 1. Let X be an MPS-space. (a) Any net which is eventually equal to an element x converges only to x. (b) If a net converges to x , then any subnet of it converges to x.

Theorem 2. Let X be an MPS-space. If addition in S is continuous at 0 , then a net converges to at most one element.

Proof. Let $\{x_n, n \in D\}$ be a net in X and let x and y be distinct elements in X. Thus, $0 \neq d(x,y) \leq$ $\leq d(x,x_n)+d(y,x_n)$. Let M be the subset of elements in S of the form $\alpha_k = \sup\{d(x,x_n): k < n \in D\}$ and let N be the subset of elements of the form $\beta_k =$ $= \sup\{d(y,x_n): k < n \in D\}$. If we assume that the given net converges to both x and y, then both M and N are directed to 0. Since addition is continuous at 0, $M+N$ is also directed to 0. Since both M and N are directed to 0, the elements of the form $\alpha_k + \beta_k$ are cofinal in $M+N$; hence, $\inf\{\alpha_k + \beta_k\} = 0$. However, the triangle inequality above shows that $0 \neq d(x,y) \leq$ $\leq \alpha_k + \beta_k$. Thus, our assumption that the given net converges to two different limits leads to a contradiction.
Q. E. D.

5. The main theorem. We now prove that the class of all MPS-spaces includes all T_1-spaces.

Theorem 3. If X is a T_1-space, then X can be regarded as an MPS-space in which convergence w.r.t. the topology is equivalent to convergence w.r.t. the PS-metric.

Proof. Let S_0 be the partially ordered semi-group consisting of two elements, 0 and 1, where $1+1=1$. Let S be the partially ordered semi-group of all functions α on pairs (U,t), where $t \in X$ and U is an open set containing t, into S_0 subject to the following condition: if there exists $t_0 \in X$ such that $\alpha(U,t_0)=0$ for all open sets U containing t_0, then $\alpha(U,t)=0$ for all pairs (U,t). In S addition and the partial ordering are defined pointwise. Although supremums in S are computed pointwise, infimums are not generally computed in this way. For any $x, y \in X$ we define $d(x, y) = \alpha \in S$, where the function α is determined as follows: $\alpha(\mathcal{U}, t) = 1$ if $x \in \mathcal{U}$ and $y \in \mathcal{U}'$ or vice versa; $\alpha(\mathcal{U}, t) = 0$ in any other case. We will now show that d is a PS-metric by verifying the necessary conditions in the order 3),1),2). We verify condition 3) first since in doing this we verify that the above definition for d is actually meaningful; this is necessary because S is not the collection of all functions on pairs $(\mathcal{U}, t)$. It is clear that $d(x, x) = 0$, the identically zero function. If $x \neq y$, then for every $t \in X$ there exists an open set $\mathcal{U}$ containing t such that $x \in \mathcal{U}$ and $y \in \mathcal{U}'$ or vice versa. This follows from the fact that X is a T_1-space. This means that if $x \neq y$, then $d(x, y)$ actually is an element in S and also that $d(x, y) \neq 0$. Thus, condition 3) is verified.

It is clear that $d(x, y) = d(y, x)$; hence, condition 1) is verified.

Let $x, y, z \in X$ and $d(x,z) = \alpha$, $d(x,y) = \beta$, $d(y,z) = \gamma$. We wish to show $\alpha \leqq \beta + \gamma$. If α is the identically zero function, then the inequality is obvious. Thus, let $(\mathcal{U}, t)$ be a pair such that $\alpha(\mathcal{U}, t) = 1$. No generality is lost in assuming $x \in \mathcal{U}$ and $z \in \mathcal{U}'$. If $y \in \mathcal{U}$, then $\gamma(\mathcal{U}, t) = 1$; if $y \in \mathcal{U}'$, then $\beta(\mathcal{U}, t) = 1$. In any event, $\alpha(\mathcal{U}, t) \leqq \beta(\mathcal{U}, t) + \gamma(\mathcal{U}, t)$ for all pairs $(\mathcal{U}, t)$. Hence, condition 2) is verified. Thus, X may be regarded as an MPS-space.

We will now show that convergence w.r.t. the topology

is equivalent to convergence w.r.t. d as defined above. Let $\{x_n, n \in D\}$ converge to x w.r.t. the topology. If $d(x, x_n) = \alpha_n \in S$, then it is clear that for every pair (U, x), where U is an open set containing x, we eventually have $\alpha_n(U, x) = 0$. Thus $\lim\sup \alpha_n(U,x) = 0$ for every open set U containing x. But by definition of S, this means that $\lim\sup d(x, x_n) = 0$, and, hence, the given net converges w.r.t. d. Now assume that the above net does not converge to x w.r.t. the topology. Then there exists an open set U containing x such that $\alpha_n(U, x) = 1$ cofinally in D. If V is defined as the complement of $\{x\}$, then V is an open set containing any $t \neq x$. Hence, for any $t \neq x$ we have $\alpha_n(V, t) = 1$ cofinally in D. Hence, $\lim\sup \alpha_n(U,x) = \lim\sup \alpha_n(V,t) = 1$ for all $t \neq x$. But this means that $\lim\sup \alpha_n \neq 0$, which means that the given net does not converge w.r.t. d.
Q. E. D.

6. A general metrization problem. Aside from the study of MPS-spaces as mathematical objects, one may use Theorem 3 as a starting point for studying the structure of topological spaces in terms of how they can be "metrized" by using various partially ordered semi-groups. In the proof of Theorem 3 we constructed a particular partially ordered semigroup in order to obtain a rather general result. However, this particular semi-group is not necessarily the "best" possible. It is here that one might formulate a "general metrization problem" in asking the question: if X is a given topological space (at least a T_1-space), then what is the "best" possible partially ordered semi-group S and the

"best" possible PS-metric making X into an equivalent MPS-space? The word "best" is used here to mean that when X is regarded as an MPS-space, then one can determine at least some topological properties of X from the properties of S and the PS-metric. We shall briefly mention a few examples to illustrate this idea.

If the metrizing semi-group S is actually linearly ordered and addition in S is continuous at 0, then X must be a Hausdorff uniform space (which may not be metrizable in the usual sense). If, in addition, we assume that $\alpha + \alpha = \alpha$ for every $\alpha \in S$, then X is totally disconnected in the sense that the open-closed sets form a basis for the topology.

As another example, let us consider the situation in which the metrizing partially ordered semi-group S has sufficiently many continuous "functionals". A functional is a non-negative real-valued function f defined on S with the properties:

a) $f(0) = 0$, b) $f(\alpha) \leqq f(\beta)$ if $\alpha \leqq \beta$, c) $f(\alpha+\beta) \leqq \leqq f(\alpha) + f(\beta)$. A functional f is said to be continuous if whenever M is a non-empty subset of S which is directed to 0, then $\inf\{f(\alpha) : \alpha \in M\} = 0$.

To say that S has sufficiently many continuous functionals simply means that if $\alpha \in S$ and $\alpha \neq 0$, then there exists a continuous functional f such that $f(\alpha) > 0$. In this present example X has the property that distinct points can be separated by a continuous real-valued function. For if y and x are distinct points in X, then $\alpha = d(y, x) \neq 0$ and, hence, there is a continuous functional f such that $f(\alpha) > 0$. From the properties of f it is easy to show that $f(d(y,\cdot))$ is a continuous real-valued function defined on X which separates the points y and x.

R e f e r e n c e s

[1] M.Ya. ANTONOVSKII, V.G. BOLTYANSKII and T.A. SARYMSAKOV: Metric spaces over semi-fields, Trudy of Tashkent State University, 1961.

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