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ON EXTREMA OF FUNCTIONALS
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Introduction. The present paper is dealing with study of extrema of functionals. One simple generalization of Vajnberg's result on existence of minimum of non-linear functional \( f \) is given and the condition for uniqueness of minimum is established. These conditions concern the second differential of \( f \). Another theorem, where the sufficient conditions (concerning gradient of the functional in question) for existence and uniqueness of extremum of \( f \) are given, is presented. Furthermore, several simple conditions for weak convergence of minimizing sequence are given and strong convergence is investigated, too.

Assuming existence of a unique minimum of the functional in question, a simple condition concerning the second differential of \( f \) is sufficient for the strong convergence of minimizing sequence. Given a sequence \( \psi_n(x) = \Phi(x) - f_n(x) \) of functionals, where \( \Phi \) is non-linear, \( f_n \) are linear (we are working in reflexive Banach spaces) and letting

\[
\psi_n(x) = \min_{x \in X} \psi_n(x) \quad \text{(this minimum existing)},
\]

\((n = 0, 1, 2, \ldots)\) the implication \( f_n \to f_0 \to x_n \to x_0 \) holds under certain conditions.

Terminology and notations used in this paper. Real Banach space is denoted by \( E \) (or \( E_X, E_Y \) etc.) - \( E^* \) is
the space of all linear and bounded functionals on $E$; the symbol $[E_x \to E_y]$ denotes the set of all linear and bounded mappings of $E_x$ to $E_y$.

Let $F$ be an operator from $E_x$ to $E_y$. We shall denote by $DF(x, h)$ linear Gateaux' differential of operator $F$ in the point $x$, i.e.

$$DF(x, h) = \lim_{t \to 0} \frac{F(x+th)-F(x)}{t}, \quad h \in E_x,$$ where

$DF(x, h)$ is bounded and linear in variable $h$. If $f$ is a functional on $E$ having a linear Gateaux' differential on the set $M \subset E$, then

$$(1) \quad Df(x, h) = F(x)h,$$ where $F(x) \in [E \to E]$, $x$ being fixed, $x \in M$.

The operator $F$ defined by the equation (1) is called gradient of the functional $f$ and we shall write

$$F(x) = \text{grad } f(x).$$

Operator $F$ defined on $E$ to $E^*$ is called potential on the set $M \subset E$, if there is such a functional $f$ that the equality

$$\text{grad } f(x) = F(x)$$
holds for all $x \in M$.

Remark 1. If the operator $F$ defined on $E$ to $E^*$ is potential on $M \subset E$, then there exists only one functional $f$, for which $f(x_0) = f_0 \quad (x_0 \text{ being a fixed point in } M)$ and $F(x) = \text{grad } f(x)$ on $M$; the functional $f$ is expressed by:

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\[ f(x) = f_o + \int_0^t F(x_0 + t(x - x_0))(x - x_0)dt \]

under certain conditions (see [1], § 5) which are fulfilled whenever this relation is used. Weak convergence is denoted by \( \overset{w}{\to} \).

\textbf{Remark 2} ([2], § 3). Banach space has a weakly compact sphere if and only if it is reflexive.

\textbf{Lemma 1} ([1], § 9). Given a Banach space \( E \) with a weakly compact sphere and given a bounded weakly closed set \( \sigma \subseteq E \) and a lower-semicontinuous functional on \( E \), then \( f \) is bounded from below on \( \sigma \) and there exists

\[
\min_{x \in \sigma} f(x).
\]

\textbf{Lemma 2}. Let \( E \) be a Banach space with a weakly compact sphere; let \( f \) be lower-semicontinuous functional on \( E \), \( x_0 \in E \) and suppose that there is a \( K > 0 \) such that \( r > K \) implies

\[
\inf_{x \in \sigma} f(x) \geq f(x_0).
\]

Then there exists an absolute minimum of \( f(x) \), i.e.

\[
\min_{x \in \xi} f(x).
\]

\textbf{Proof}. Let \( r_0 = \max(\|x\|, \|x_0\|); D_{r_0} = \{x; \|x\| \leq r_0\} \).

There exists \( \min_{x \in D_{r_0}} f(x) \) according to Lemma 1. Now it is trivial to show that

\[
\min_{x \in \xi} f(x) = \min_{x \in D_{r_0}} f(x).
\]

\textbf{Definition}. A point \( x_0 \) is a critical point of the functional \( f \) if

\[
\text{grad } f(x_0) = \theta, \quad (\|\theta\| = 0).
\]

\textbf{Theorem 1}. Let \( E \) be a Banach space with a weakly compact sphere. Assume that: 1) The functional \( f \) has
Gateaux' differential of the first and the second orders on \( E \) and the inequality

\[
D^2 f(x, h, h) > \gamma(\|h\|) \cdot \|h\|
\]

holds for all \( h \in E \), where \( \gamma(t) \) is a continuous, real-valued function on \((0, +\infty)\), non-negative such that

\[
\lim_{R \to \infty} \frac{1}{R} \int_0^R \gamma(t) dt = \infty.
\]

2) \( D^2 f(tx, h, h) \) is continuous for \( t \in (0, 1) \).

Then there exists \( \min_{x \in E} \frac{f(x)}{x} \). Furthermore, if \( \gamma(t) > 0 \) for \( t > 0 \), then there exists only one extremal point.

**Proof.** The first assumption implies the lower-semi-continuity of functional \( f \) in any sphere in \( E \). According to Lemmas 1 and 2 it is sufficient to show that there exists a number \( R_0 > 0 \) such that for \( R > R_0 \) the inequality

\[
\inf_{x \in \mathbb{R}} f(x) > f(x_0)
\]

holds (\( x_0 \) is a point in the sphere \( \{ x ; \|x\| \leq R_0 \} \)). Let \( F(x) = \text{grad} f(x) \). Then, according to (2), we can write

\[
F(x)h = F(\theta)h + \int_0^1 DF(tx, x)h dt
\]

particularly, for \( h = x \) we have

\[
F(x)x = F(\theta)x + \int_0^1 DF(tx, x)x dx \geq F(\theta)x + \|x\| \cdot \gamma(\|x\|).
\]

Consequently, the relation

\[
f(x) = f(\theta) + \int_0^1 F(tx)tx dt
\]

implies the estimate

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on the sphere \( \| x \| = R \), or \( f(x) > f(\theta) + R \cdot \int_0^1 \gamma(tR) \, dt \).

But \( \int_0^1 \gamma(tR) \, dt = \frac{1}{R^4} \int_0^1 \gamma(t) \, dt \), so that for a given \( K > 0 \) there exists a number \( R_0 \) such that for \( R > R_0 \) the inequality \( f(x) > f(\theta) + K \) holds on the sphere \( \| x \| = R_0 \), i.e. \( \inf_{\| x \| = R_0} f(x) > f(\theta) \).

The second part of the theorem is trivial. If both \( x_1 \) and \( x_2 \) are critical points and \( x_1 - x_2 \neq 0 \), we have

\[
0 = Df(x_1, h) - Df(x_2, h) = Df(x_1 + \tau(x_2 - x_1), h, x_2 - x_1)
\]

for all \( h \in E \); especially for \( h = x_2 - x_1 \) we have a contradiction.

Remark 3 ([1], § 9). If \( x_0 \) is an extremal point of \( f \) on the open set \( \omega \subset E \) and there exists \( Df(x_0, h) \), then the point \( x_0 \) is critical.

Theorem 2. Let \( E \) be a Banach space with the weakly compact sphere; \( F \) potential operator on \( E \) to \( E^* \); \( x_0 \in E \) and let \( F(x_0 + t(x - x_0)) (x - x_0) \) be continuous for \( t \in (0, 1) \). Assume that there exists a measurable function \( \lambda_{x_0}(t) \), defined on \( (0, \infty) \) such that

a) \( \lambda_{x_0}(t) \) is bounded on any finite interval;

b) there exists \( R_0 \) such that \( \int_0^{R_0} \frac{\lambda_{x_0}(t)}{t} \, dt > 0 \);

c) \( F(x)(x - x_0) > \lambda_{x_0}(\| x - x_0 \|) \);

d) \( x_n \xrightarrow{\omega} x \Rightarrow F(x)(x - x_0) \leq \lim F(x_n)(x_n - x_n) \).

Let us designate by \( f(x) \) the functional for which
\[ F(x) = \text{grad } f(x). \]

I. Then there exists a local minimum of the functional \( f \) and accordingly a critical point.

II. Furthermore, if
\[ \int_{0}^{R} \frac{\partial \phi}{\partial \nu} \, ds > 0 \quad \text{for} \quad R > R_0 \]
then there exists an absolute minimum of \( \phi \).

III. Furthermore, if
\[ \int_{0}^{R} \frac{\partial \phi}{\partial \nu} \, ds > 0 \quad \text{for} \quad R > 0 , \]
then the absolute minimum is unique.

IV. If, for arbitrary points \( x_1, x_2 \in E \); \( x_1 \neq x_2 \)
\[ (F(x_1) - F(x_2))(x_1 - x_2) > 0 \]
then \( f \) has at most one critical point.

**Proof.** We shall prove that \( f \) is weakly lower-semicontinuous on \( E \). The first assertion then follows from Lemma 1 and the fact that \( f(x) > f(x_o) \) for \( x, \|x - x_o\| = R \).
(According to Lemma 1 there exists \( \min_{\|x - x_o\| = R} f(x) \) and as a result of the relation \( f(x) > f(x_o) \) on \( \|x - x_o\| = R \) there exists a critical point.)

Let \( x_n, x^o \in E \); \( x_n \xrightarrow{w} x^o \). The inequality
\[ F(x_n + t(x - x_o))(x - x_o) \geq \frac{\lambda_{x_n}(t \cdot \|x - x_o\|)}{t \cdot \|x - x_o\|} \cdot \|x - x_o\| \]
holds on the assumption (c) \( t \) is positive). Because of boundedness of \( \|x_n - x_o\| \) \( \|x_n - x_o\| \) is bounded owing to weak convergence of \( \{x_n\} \) we have according to (a)

\[ (5) F(x_o + t(x_n - x_o))(x_n - x_o) > -M; M > 0, t \in (0, 1); n = 1, 2, ... . \]
Now,
\[ \int_0^1 F(x_0 + t(x - x_0)) (x - x_0) dt \leq \lim_{\epsilon \to 0} \int_0^1 F(x_0 + t(x - x_0)) (x - x_0) dt \leq \lim_{\epsilon \to 0} \int_0^1 F(x_0 + t(x - x_0)) (x - x_0) dt, \]
where the last inequality follows from Fatou's lemma which can be used according to (5). Now, using the relation
\[ f'(x) = f(x_0) + \int_0^1 F(x_0 + t(x - x_0)) (x - x_0) dt \]
and applying the inequality (6), we obtain the desired result.

II The second statement trivially follows from Lemma 2 and the assumption (e) using the fact
\[ f(x) = f(x_0) + \int_0^1 F(x_0 + t(x - x_0)) (x - x_0) dt. \]

III Let \( f(x_1) = f(x_2) \) be minimum of \( f(x) \); \( x_1 \neq x_2 \).

Then we have
\[ f(x_2) = f(x_1) + \int_0^1 F(x_1 + t(x_2 - x_1)) (x_2 - x_1) dt \]
so that the following relation must hold
\[ 0 = \int_0^1 F(x_1 + t(x_2 - x_1)) (x_2 - x_1) dt \equiv \int_0^1 \lambda \cdot (t \cdot |x_2 - x_1|) \frac{dt}{t} = \int_0^1 \frac{\lambda(x_2)}{h} ds > 0, \]
which is a contradiction.

IV Let \( x_1 \neq x_2 \); \( x_1, x_2 \in E \);
\[ \text{grad } f(x_1) = F(x_1) = 0; \quad \text{grad } f(x_2) = F(x_2) = 0. \]
Then we have
\[ 0 = (F(x_1) - F(x_2)) (x_1 - x_2) > 0; \]
i.e. a contradiction.

**Theorem 3.** Let \( E \) be a space with a weakly compact sphere, \( f(x) \) weakly lower-semicontinuous functional on \( E \), \( x_0 \) point of the local minimum of \( f \) such that there exists \( \kappa > 0 \) such that for \( \{ x; 0 < \| x - x_0 \| < \kappa \} \)
the relation \( f(x) > f(x_0) \) holds. Let \( \| x_n - x_0 \| \leq \kappa \),

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\[ f(x_n) \to f(x_0). \text{ Then } x_n \overset{w}{\to} x_0. \]

**Proof.** Let us suppose the contrary. The sequence \{\(x_n\)\} is bounded so that there is a subsequence \(\{x_{n_k}\}\); \(x_{n_k} \overset{w}{\to} \hat{x} \neq x_0\). Then we have

\[ f(\hat{x}) = \lim f(x_{n_k}) = f(x_0) \implies f(\hat{x}) = f(x_0), \]

which is a contradiction.

**Theorem 4.** Let \(E\) and \(f\) be defined just as in Theorem 3. Let \(\lim_{\|x\| \to \infty} f(x) = \infty\) and let there be a unique minimum of \(f\); let us denote it \(f(x_0)\). Then the implication \(f(x_n) \to f(x_0) \implies x_n \overset{w}{\to} x_0\) holds.

**Proof.** It is clear that \(f(x) > f(x_0)\) for \(x \in E - \{x_0\}\). Let \(\{x_n\}\) be such a sequence that \(f(x_n) \to f(x_0)\).

Either \(\{x_n\}\) is bounded and the assertion follows from Theorem 3 or \(\|x_n\| \to \infty\) but in this case the assumption \(f(x_n) \to f(x_0)\) does not hold.

**Theorem 5.** Let \(E\) be a Banach space with a weakly compact sphere; \(f(x)\) functional on \(E\) which satisfies all the conditions of Theorem 1 so that there is \(\min_{x \in E} f(x) = f(x_0) = c\). Let \(\{x_n\}\) be minimizing sequence, i.e.

\(f(x_n) \to f(x_0)\). Let there be a number \(c > 0\) and a point \(t_0 \in (0, \infty)\) such that for \(t > t_0\) the inequality \(\gamma(t) > c\) holds. Then \(x_n \to x_0\).

**Proof.** Let

\[ \phi(x, y) = \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2}\right). \]

We shall arrange the expression on the right-hand side using formula (2) and Fubini's theorem:
\[ g(x, y) = \frac{1}{2} \left[ f(x) - f\left(\frac{x + y}{2}\right) \right] + \frac{1}{2} \left[ f(y) - f\left(\frac{x + y}{2}\right) \right] = \]

\[ = \frac{1}{4} \int_0^1 \frac{d}{dt} \left( \frac{x + y}{2} + t \cdot \frac{y - x}{2}, x - y \right) dt + \frac{1}{4} \int_0^1 \frac{d}{dt} \left( \frac{x + y}{2} + t \cdot \frac{y - x}{2}, x - y \right) dt = \]

\[ = \frac{1}{4} \int_0^1 \left[ \frac{d}{dt} f\left(\frac{x + y}{2} + t \cdot \frac{y - x}{2}, x - y\right) - \frac{d}{dt} f\left(\frac{x + y}{2}, x - y\right) \right] dt = \]

\[ = \frac{1}{4} \int_0^1 dt \int_0^1 \left[ \frac{d}{dt} f\left(\frac{x + y}{2} + t \cdot \frac{y - x}{2}, x - y\right) - f\left(\frac{x + y}{2}, x - y\right) \right] ds. \]

Using the first assumption in Theorem 1 we obtain
\[ g(x, y) \geq \frac{1}{8} \int_0^1 \gamma (\|x - y\|) \cdot \|x - y\| dt = \frac{1}{8} \gamma (\|x - y\|) \cdot \|x - y\|. \]

Further, \( \varepsilon > 0 \) being arbitrary, there exists \( n_0 \) such that \( f(x_{n}) \leq d + \varepsilon \) for all \( n \geq n_0 \). Then the following relation holds:
\[ g(x_{n}, x_{0}) \leq \frac{d + \varepsilon}{2} + \frac{d}{2} - d < \varepsilon. \]

Choosing \( \varepsilon = \frac{\varepsilon}{8} \) we have proved that for arbitrary
\[ \varepsilon > 0 \] there exists \( n_0 > 0 \) such that for \( n \geq n_0 \) the relation
\[ \gamma (\|x_{n} - x_{0}\|) \cdot \|x_{n} - x_{0}\| < \varepsilon \]
holds.

Now, the minimizing sequence \( \{x_{n}\} \) is bounded by the last assumption of the theorem in question (one can prove it easily by contradiction); let \( \|x_{n} - x_{0}\| \leq K < \infty \). If we can choose a subsequence \( \{x_{n_{k}}\} \) of \( \{x_{n}\} \) such that for some \( \varepsilon_{0} > 0 \) the relation \( \|x_{n_{k}} - x_{0}\| > \varepsilon \)
holds,
then
\[ \lim \sup \gamma (\|x_n - x_0\|) \cdot \|x_n - x_0\| \geq \min_{\epsilon \in (0,K)} \gamma (\epsilon) \cdot \epsilon > 0; \]
this is in contradiction with (7).

**Lemma 3.** Let \( E \) be a Banach space with a weakly compact sphere; \( \Phi (x) \) is a non-linear functional on \( E \). Let \( \psi_f(x) = \phi(x) - f(x) \) for an arbitrary linear functional \( f \) on \( E \). Given a positive number \( K_1 \) let \( \psi_f(x) \) satisfy the conditions of Theorem 1 for those \( f \) for which \( \|f\| \leq K_1 \). Let us denote \( \min_{x \in E} \psi_f(x) \) by \( \psi_f(x_f) \). Then there is a positive number \( K_2 \) (depending on \( K_1 \)) such that \( \|x_f\| \leq K_2 \).

**Proof.** In the first part of the proof of Theorem 1 we obtained the estimate
\[ \psi_f(x) \geq \psi_f(\theta) + R \cdot \|F_f(\theta)\| + \int_0^R \gamma (t R) \, dt, \]
where \( F_f(x) = \text{grad} \, \psi_f(x) \).
Here we have \( F_f(x) = \text{grad} \, \phi(x) - f = \Phi(x) - f \).
From (8) it follows
\[ \phi(x) \geq \phi(\theta) + R \cdot \|\Phi(\theta)\| - 2 K_1 \int_0^R \gamma (t R) \, dt; \]
according to this inequality there exists \( R_o > 0 \) such that for \( R > R_o \) the relation \( \phi(x) > \phi(\theta) \) holds.
Now it can be shown clearly that for arbitrary \( K_2 > R_o \) there is \( \|x_f\| \leq K_2 \). Actually, if \( \|f\| \leq K_1 \), we obtain from (8)
\[ \psi_f(x) - K_1 > \psi_f(\theta) - \|\Phi(\theta)\| - K_1 + \int_0^R \gamma (t R) \, dt \]
and \( \|\Phi(\theta)\| - K_1 \geq - \|\Phi(\theta)\| - 2 K_1 \), so that the inequality \( \psi_f(x) > \psi_f(\theta) \) holds on the sphere \( \|x\| = R \geq R_1 \)
(where \( R_1 \) is a number, \( R_1 \leq R_o \)) and the point of \( \min \psi_f(x) \) cannot be contained outside of sphere.
Remark 4. Roughly speaking, if the functionals $f$ are in a fixed sphere then there is a fixed sphere which contains all the points of $\min \psi_i(x)$ (under certain conditions).

Theorem 6. Let $E$ be a Banach space with a weakly compact sphere, let $\Phi$ be a non-linear functional on $E$. Let $f_i (i = 0, 1, 2, \ldots)$ be linear functionals on $E$, $f_n \to f_0$ (in $E^*$) $(n = 1, 2, \ldots)$. Let us write $\psi_i(x) = \Phi(x) - f_i(x)$. Let $\psi_i(x)$ satisfy the conditions of Theorem 1. Let $\psi_i(x_i) = \min_{x \in E} \psi_i(x)$. Then $x_n \to x_0$ in $E$.

Proof. $x_i$ is an extremal point of functional $\psi_i(x)$ so that $\nabla \psi_i(x_i) = 0$, i.e.

$$0 = \nabla \psi_n(x_n) = \nabla \Phi(x_n) - f_n (n = 0, 1, 2, \ldots).$$

From this fact it follows

$$\|\nabla \Phi(x_n) - \nabla \Phi(x_0)\| \leq \|f_n - f_0\| \to 0,$$

and further

$$(9) \|\nabla \Phi(x_n) - \nabla \Phi(x_0)\| \leq \|f_n - f_0\| (n \to \infty) \to 0.$$ 

It is $\nabla \Phi(x_0) = \partial \Phi(x_0, h)$ for $h \in E$. Let $h_i = x_i - x_0$. Because of $f_n \to f_0$ there is a positive number $K_1$ such that $\|f_i\| \leq K_1$ and, according to Lemma 3, there is a number $K_2 > 0$ such that $\|x_i\| \leq K_2$, so that $\|h_i\| \leq K$. Now, according to Remark 1, we have

$$D\Phi(x_n, h_n) - D\Phi(x_0, h_0) = \int_0^1 \partial^2 \Phi(x_0 + t(x_n - x_0), h_n, x_n - x_0)dt$$

and if $h_n = x_n - x_0$ we obtain

$$D\Phi(x_n, h_n) - D\Phi(x_0, h_0) \geq \gamma(\|h_n\|, \|h_n\|).$$

Let $\epsilon$ be an arbitrary positive number. Now, for $\varepsilon_1 = \frac{\epsilon}{K}$ there is $n_0 > 0$ such that for $n > n_0$ the following
relation holds (according to (9)):
\[ \gamma (\| h_n \| \cdot \| h_n \| \leq \| \nabla \phi (x_n) - \nabla \phi (x) \| \cdot \| h_n \| < \epsilon, K = \epsilon \]
so that we have proved:
for arbitrary \( \epsilon > 0 \) there exists \( n_\epsilon > 0 \) such that for \( m > n_\epsilon \) the following relation holds: \( \gamma (\| x_n - x_\epsilon \| \cdot \| x_n - x_\epsilon \| < \epsilon \). Now, as in Theorem 5, we obtain \( x_n \rightarrow x_\epsilon \).

Remark. After the paper was submitted the authors became aware that Theorem 1 is stated in "M.M. Vajnberg: O minimume vypuklyh funkcionalov, UMN 20(1965),121,No.1, 239-240" without proof.

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