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COORDINATIZATION OF PARALLEL SYSTEMS, II x)

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In this part we shall use ternary halfgroupoids for the coordinatization of certain "parallel systems". Further we shall investigate as a special case some systems very closed to pseudo planes in the sense of Sandler ([3],p.301).

1. In the following, it is necessary to distinguish between partitions in a set and partitions on a set: A <u>partition in</u> (<u>on</u>) a nonempty set S is a nonempty set of nonempty subsets in S which are pairwise disjoint (which are pairwise disjoint and cover S).

Now we generalize somewhat the definition of a parallel system used in Part I: By a "<u>parallel system</u>" $\mathcal{P}$ we shall mean a triplet $(\mathcal{R}, \mathcal{L}, \parallel)$ where (i) $\mathcal{R}$ is a nonempty set of elements called the <u>points</u>, (ii) $\mathcal{L}$ is a nonempty set of some nonempty subsets in $\mathcal{R}$ called the <u>lines</u> and (iii) $\parallel$ is a partition on $\mathcal{L}$ such that each member of $\parallel$ is a partition in $\mathcal{R}$.

2. Two parallel systems $\mathcal{P} = (\mathcal{R}, \mathcal{L}, \parallel)$, $\mathcal{P}' = (\mathcal{R}', \mathcal{L}', \parallel')$ are said to be <u>isomorphic</u> if there is a bijective mapping

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x) Part I: Comment.Math.Univ.Carolinae 7,3(1966),pp.325-333

$\rho : \mathcal{P} \to \mathcal{P}'$ such that

(1) $\rho l \in \mathcal{L}'$ for each $l \in \mathcal{L}$, and

(2) $\rho l, \rho m$ belong to the same member of $/\!/'$ if l, m belong to the same member of $/\!/$.

3. A <u>ternary halfgroupoid</u> T is defined as a couple (S, τ) where S is a nonempty set and τ a mapping of a nonempty set $\mathcal{D}om\ \tau \subseteq S \times S \times S$ into S. For $\mathcal{D}om\ \tau = S \times S \times S$ we get a <u>ternary groupoid</u> (called also a <u>ternary ring</u>).

Denote by $(\mathcal{D}om\ \tau)_{ij}$ and $(\mathcal{D}om\ \tau)_k$ the projection of $\mathcal{D}om\ \tau$ obtained by leaving only the i-th and the j-th component or leaving only the k-th component respectively. For each $(u, v) \in (\mathcal{D}om\ \tau)_{23}$, define $L(u, v)$ as a nonempty set $\{(x, y) \mid y = \tau(x, u, v)\}$, and, for each $u \in (\mathcal{D}om\ \tau)_2$, define $\mathbb{L}(u)$ as a set consisting of members $L(u, v)$ where v runs over all values such that $(u, v) \in (\mathcal{D}om\ \tau)_{23}$.

4. We shall use two following conditions for a ternary halfgroupoid $T = (S, \tau)$:

(3) $\tau(a, u, v_1) = \tau(a, u, v_2)$ for (a, u, v_1), $(a, u, v_2) \in \mathcal{D}om\ \tau \Longrightarrow v_1 = v_2$;

(4) $\tau(x, u_1, v_1) = \tau(x, u_2, v_2)$ for (u_1, v_1), $(u_2, v_2) \in (\mathcal{D}om\ \tau)_{23}$ such that $\{x \in S \mid (x, u_1, v_1) \in \mathcal{D}om\ \tau\} = \{x \in S \mid (x, u_2, v_2) \in \mathcal{D}om\ \tau\}$, identically in $x \Longrightarrow (u_1, v_1) = (u_2, v_2)$.

5. Two ternary halfgroupoids $T = (S, \tau)$, $T' = (S', \tau')$ are said to be <u>isomorphic</u> if there is a bijective mapping $\sigma : S \to S'$ such that

(5) $\{(\sigma x, \sigma u, \sigma v) \mid (x, u, v) \in \mathcal{D}om\ \tau\} = \mathcal{D}om\ \tau'$,

(6) $\tau'(\sigma x, \sigma u, \sigma v) = \sigma\tau(x, u, v)$ for all $(x, u, v) \in$ $\in \mathcal{D}om\ \tau$.

6. Let $T = (S, \tau)$ be a ternary halfgroupoid satisfying (3) and (4). Set $\mathcal{P} = \bigcup_{(u,v) \in (\mathcal{D}om\ \tau)_{23}} L(u, v)$,

$\mathcal{L} = \{L(u, v) \mid (u, v) \in (\mathcal{D}om\ \tau)_{23}\}$ and $/\!/ = \{\mathbb{L}(u) \mid u \in (\mathcal{D}om\ \tau)_2\}$ where, for each $u \in (\mathcal{D}om\ \tau)_2$, $\mathbb{L}(u)$ consists of $L(u, v)$ such that $(u, v) \in (\mathcal{D}om\ \tau)_{23}$. By (3), each $\mathbb{L}(u)$ consists of mutually disjoint nonempty members. By (4), any two $L(u_1, v_1)$, $L(u_2, v_2)$ with (u_1, v_1), $(u_2, v_2) \in (\mathcal{D}om\ \tau)_{23}$, $u_1 \neq u_2$, must be distinct so that $\{\mathbb{L}(u) \mid u \in (\mathcal{D}om\ \tau)_2\}$ is a partition on $\mathcal{L}$. Thus, $(\mathcal{P}, \mathcal{L}, /\!/)$ is a parallel system (called <u>associated</u> to T). Obviously, the parallel system associated to T is determined canonically.

Define $Y(a) = \{(x, y) \in S \times S \mid x = a\}$, $X(a) = \{(x, y) \in S \times S \mid y = a\}$ for all $a \in S$, and notice that, in the preceding, it must hold $\operatorname{card}(Y(a) \cap \cap\, l) \leq 1$ for all $a \in S$ and all $l \in \mathcal{L}$.

7. Let $\mathscr{P} = (\mathcal{P}, \mathcal{L}, /\!/)$ be a parallel system. Let there exist a set S and an injective mapping $\alpha : \mathcal{P} \to S \times S$ such that

(7) $card(\gamma(a) \cap \alpha l) \leq 1$ for all $a \in S$, $l \in \mathcal{L}$,

(8) $card \, /\!/ \leq card \, S$.

Choose an injective mapping $\beta : /\!/ \rightarrow S$ and, for each $L \in /\!/$, an injective mapping $\gamma_L : L \rightarrow S$. Finally, define a mapping τ of a certain subset of $S \times S \times S$ into S as follows: $y = \tau(x, u, v) \Longleftrightarrow \alpha^{-1}(x, y) = P \in \gamma_L^{-1} v$ where $\beta L = u \in \beta /\!/$ and $v \in \gamma_L L$. By the preceding assumptions, τ must be single-valued and is well defined. So $T = (S, \tau)$ is a ternary halfgroupoid (called associated to $\mathcal{P}$ with respect to α, β, γ_L).

8. Let $T = (S, \tau)$ be a ternary halfgroupoid satisfying (3) and (4). Let $\mathcal{P} = (\mathcal{P}, \mathcal{L}, /\!/)$ by its associated parallel system. The next conditions are equivalent:

(9) Each $L \in /\!/$ is a partition on $\mathcal{P}$.

(9) The equation $\tau(a, c, v) = b$ has a unique solution $v \in (Dom\,\tau)_3$ for any $u \in (Dom\,\tau)_2$ and $(a, b) \in \mathcal{P}$.

For the proof it suffices to note that (9_2) says precisely that, in every $L \in /\!/$, there is exactly one line of $\mathcal{P}$ passing through any point of $\mathcal{P}$.

9. A "projective" pseudo-plane can be defined (cf. [3], p.301) as a triplet $(\mathcal{P}, \mathcal{L}, I)$ where $\mathcal{P}, \mathcal{L}$ are sets (of points and lines, respectively) and I is an incidence relation (i.g. $I \subseteq \mathcal{P} \times \mathcal{L}$ s.t. $A_i \, I \, a_j$ for $i, j = 1, 2$ implies $A_1 = A_2$ or $a_1 = a_2$) such that there exist points $P_1 \neq P_2$ and lines $l_1 \neq l_2$ with

$P_1, P_2 \, I \, l_1 \,;\; P_1 \, I \, l_2$ for which the following conditions hold: (i) For any point P such that $P \, I \, l_1$ or $P \, I \, l_2$ and any point $Q \neq P$ there is a unique line l with $P, Q \, I \, l$. (ii) For any line l such that $P_1 \, I \, l$ or $P_2 \, I \, l$ and any line m with $P_1 \, \not{I} \, m$ or $P_2 \, \not{I} \, m$ there is a unique point P with $P \, I \, l, m$. (iii) There are four points no three of which are incident with the same line. - If l_1 and all points incident with l_1 are deleted then one obtains an <u>"affine" pseudo-plane.</u> We shall show that such affine pseudo-planes can be introduced in another way.

10. A parallel system $\mathcal{P} = (\mathfrak{P}, \mathcal{L}, /\!/)$ will be called an <u>almost pseudo-plane</u> if it satisfies (9_1) and

(10) $\mathfrak{P} = S \times S$ for a set S containing at least two distinct elements;

(11_1) $Y(a) \in \mathcal{L}$ for each $a \in S$; x)

(11_2) $X(a) \in \mathcal{L}$ for each $a \in S$;

(12_1) $\mathbf{Y} = \{Y(a) \mid a \in S\} \in /\!/$;

(12_2) $\mathbf{X} = \{X(a) \mid a \in S\} \in /\!/$;

(13_1) $\mathrm{card}\,(Y(a) \cap l) = 1$ for all $a \in S, l \in \mathcal{L} \setminus \mathbf{Y}$;

(13_2) $\mathrm{card}\,(X(a) \cap l) = 1$ for all $a \in S, l \in \mathcal{L} \setminus \mathbf{X}$;

(14) there is a line $Y \in \mathbf{Y}$ such that $\mathrm{card}\,(Y \cap l) = 1$ for each $l \in \mathcal{L} \setminus \mathbf{Y}$ and

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x) Cf. the definition of $Y(a)$ and $X(a)$ in Nr.6.

(15) there is bijective mapping $\beta : /\!/ \setminus \{\mathbb{Y}\} \to S$ with $\beta X = 0$ where $0 \in S$ is determined by $Y(0) = \mathbb{Y}$.

11. Let $\mathcal{P} = (\mathcal{P}, \mathcal{L}, /\!/)$ be an almost pseudo-plane. Take a parallel system $\mathcal{P}^* = (\mathcal{P}, \mathcal{L} \setminus \mathbb{Y}, /\!/ \setminus \{\mathbb{Y}\})$ and choose $\alpha = id$, β as in Nr.10 and, for every $\mathbb{L} \in /\!/ \setminus \{\mathbb{Y}\}$, $\gamma_{\mathbb{L}} : \mathbb{L} \to S$ determined by $\gamma_{\mathbb{L}}\, l$ to be equal to the second component of the common point of l, Y (the existence of such a point is guaranteed by (9_1) and (14)). Let $T = (S, \tau)$ be the ternary groupoid associated to $\mathcal{P}^*$ with respect to $\alpha, \beta, \gamma_{\mathbb{L}}$. It can be verified that T satisfies the conditions $card\ S \geq 2$, (4) and

(16) $\tau(x, 0, v) = \tau(0, u, v) = v$ for all $x, u, v \in S$;

(17) for any $x, u, v \in S$ there is a unique $v \in S$ such that $y = \tau(x, u, v)$;

(18) for any $y, v \in S$ and $u \in S \setminus \{0\}$ there is a unique $x \in S$ such that $y = \tau(x, u, v)$.

$\mathcal{P}$ becomes a pseudo-plane if and only if

(19_1) any $P \in Y$ and any $Q \in \mathcal{P} \setminus Y$ are contained in exactly one common line of $\mathcal{P}$.

(19_1) is equivalent with its algebraic counterpart ;

(19_2) for any $x \in S \setminus \{0\}$ and $y, v \in S$ there is a unique $u \in S$ such that $y = \tau(x, u, v)$.

Conversely, it may be proved that for a ternary groupoid $T = (S, \tau)$ satisfying $card\ S \geq 2$, (4),(16),(17),(18) or $card\ S \geq 2$,(4),(16),(17),(18),(19), the associated parallel system $(\mathcal{P}, \mathcal{L}, /\!/)$ leads to the parallel system $(\mathcal{P}, \mathcal{L} \cup \mathbb{Y}, /\!/ \cup \{\mathbb{Y}\})$ which is an almost pseudo-plane or a pseudo-plane respectively.

So the preceding two types of ternary groupoids may be termed as almost pseudo-planar and pseudo-planar respectively.

Note that the pseudo-planar ternary groupoids have a more general structure as "pseudoternaries" ([3],p.303) because the existence of unit element is not required.

12. Let $\mathcal{P} = (\mathcal{P}, \mathcal{L}, /\!/)$ be an almost pseudo-plane. Suppose that it contains a diagonal line d characterized by

(20) $d = \{(x, y) \mid x = y\}$.

Let T be associated to $\mathcal{P}^*$ as in Nr.11. Then, by the immediate translation from the geometric into the algebraic language (and conversely), it may be shown that $(21_1) \Longleftrightarrow (21_2)$ where:

(21_1) Let $A_1, A_2, A_3, B_1, B_2, B_3$ be points satisfying a) $A_1 = (0, 0)$, b) there are mutually distinct lines $l_1, l_2, l_3 \in \mathcal{Y}$ such that $A_1, B_1 \in l_1$; $A_2, B_2 \in l_2$; $A_3, B_3 \in l_3$; c) there are lines a_3, b_3 from the same member of $/\!/$ such that $A_1, A_2 \in a_3$; $B_1, B_2 \in b_3$; d) there are lines a_2, b_2 belonging together with d to the same member of $/\!/$ such that $A_1, A_3 \in a_2$ and $B_1, B_3 \in b_2$ and e) A_2, A_3 lie on the same line of $\mathcal{X}$. Then B_2, B_3 lie on the same line of $\mathcal{X}$.

(21_2) $\tau(\tau(x,u,0),e,v) = \tau(x,u,v)$ for all $x, u, v \in S$ where e is determined by $d \in \mathbb{L}(e)$.

(21_2) is called the linearity condition. Cf. theorem 12 in [3], p.311 where moreover (19_1) is postulated. The derived compositions $\underset{e}{+}$, $\cdot$ (defined by $x \underset{e}{+} v = \tau(x, e, v)$, $x \cdot u = \tau(x, u, v)$ respectively) are associative if and only if the corresponding Reidemeister configuration conditions known from the web theory are satisfied. There is a very closed connection between 4-webs ([1],pp.61-63) and pseudo-planes: pseudo-planes are only certain natural "extensions" of 4-webs.

13. The construction of almost pseudo-planar ternary groupoids with linearity condition can be given as follows: Take a loop $\mathcal{L} = (S, +)$ with $card\ S \geq 2$ and choose an injective mapping $\varkappa : S \setminus \{0\} \to \mathcal{F}$ where $\mathcal{F}$ denotes the set of all permutations of S reproducing the element 0. Further, let $\varkappa 0$ be the mapping which sends each $a \in S$ onto 0. Now define the multiplication $\cdot$ by $x \cdot u = (\varkappa u)x$ for all $x, u \in S$ and the ternary composition $\tau : S \times S \times S \to S$ by $\tau(x, u, v) = x \cdot u + v$ for all $x, u, v \in S$. Then each of the conditions (4), (16), (17), (18) is fulfilled and the obtained ternary groupoid $T = (S, \tau)$ must be almost pseudo-planar. (Cf. the general principle for the construction of double groupoids over a given groupoid given in [4],pp.67-68).

In particular, if $card\ S = 3$ then there are only two distinct permutations reproducing 0 and the resulting T is necessarily planar (i.e., the associated parallel system leads to an affine plane). If $card\ S > 3$ then it is possible to choose $\varkappa$ in such a way that $\varkappa(S \setminus \{0\})$ does not act simply transitively on $S \setminus \{0\}$. Thus there exist almost pseudo-planar ternary groupoids which are not pseudo-planar.

14. Be given ternary halfgroupoids $T = (S, \tau)$ and $T' = (S', \tau')$ with associated parallel systems $\mathcal{P} = (\mathcal{P}, \mathcal{L}, /\!/)$ and $\mathcal{P}' = (\mathcal{P}', \mathcal{L}', /\!/')$ respectively. Any isomorphism between T and T$'$ induces an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$.

Proof. Let $\sigma : S \rightarrow S'$ be a bijective mapping determining the given isomorphism between T and T$'$. Let $l = \{(x, y) \mid y = \tau(x, u, v)\}$ for $(u, v) \in (\mathcal{D}om\ \tau)_{23}$. If $(x, y) \in l$ then, by (5) and (6), $\sigma y = \tau'(\sigma x, \sigma u, \sigma v)$ and by the bijectivity of σ, $\sigma l \in \mathcal{L}'$ and (1) is fulfilled. Similarly for (2).

We finish this paper with one remark about affinities of parallel systems. An isomorphism of a parallel system $\mathcal{P} = (\mathcal{P}, \mathcal{L}, /\!/)$ onto $\mathcal{P}$ may be called an <u>affinity</u> of $\mathcal{P}$. A <u>translation</u> of $\mathcal{P}$ is an affinity σ of $\mathcal{S}$ having the following property: l and σl belong to the same member of $/\!/$ for each $l \in \mathcal{L}$. A translation σ of $\mathcal{P}$ may be termed <u>central</u> if there is a $\mathfrak{C} \in /\!/$ such that $\sigma l = l$ for all $l \in \mathfrak{C}$. Some properties of central translations of groups with a partition are found in [2],pp.94-98 and 158-160. Certain similar results on central translations of groupoids with a parallelisable partition are contained in [5], but no results about central translations of general parallel systems are known to the author.

R e f e r e n c e s

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