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LINEAR DIFFERENTIAL EQUATIONS WITH NEWTON-INTEGRABLE COEFFICIENTS

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Summary: Existence theorems (items 8,9 and 11) and a unicity theorem (item 12) are given for the linear differential equation $x' = A(t)x + b(t)$ in n -space under the assumption that $A(t)$ and $b(t)$ are Newton-integrable, i.e., are exact derivatives on an interval.

1. Introduction. The classical notion of a solution of a differential equation

$$x' = f(t, x) \tag{1.1}$$

with continuous right-hand side was generalized by Carathéodory [1] as follows. We say that an absolutely continuous function φ defined on an interval $I = [\tau, \tau + \alpha]$, $\alpha > 0$, is a solution of (1.1) iff

$$\varphi'(t) = f(t, \varphi(t)) \tag{1.2}$$

a.e. on I. On setting $\varphi(\tau) = \xi$, we have $\varphi(t) = \xi + \int_\tau^t f(s, \varphi(s))\, ds$, with the integration taken in the sense of Lebesgue.

In [3] it is shown that, in such situations, several integration processes more general than that of Lebesgue may be used, thus providing a more complete theory. It is then natural to inquire into the usefulness, in this context, of another familiar integration process, namely that of Newton integration [4].

2. In what follows, n is a fixed positive integer, $\mathcal{R}^n$ denotes euclidean n-space, $\mathcal{R} = \mathcal{R}^1$. By the (t, x)-space we mean $\mathcal{R}^{n+1}$, the generic point of which may be denoted as (t, x) or $(t, x_1, \dots, x_n)$. For $x = (x_1, \dots, x_n) \in \mathcal{R}^n$ set $|x| = \max(|x_1|, \dots, |x_n|)$. For any subset $\mathcal{D}$ of the (t, x)-space, and any $x \in \mathcal{R}^n$, we denote $\mathcal{D}^{(\cdot, x)} = \{t \in \mathcal{R}; (t, x) \in \mathcal{D}\}$, and similarly for $\mathcal{D}^{(t, \cdot)}$; finally, $proj_t \mathcal{D} = \bigcup \{\mathcal{D}^{(\cdot, x)}; x \in \mathcal{R}^n\}$, and similarly $proj_x \mathcal{D} = \bigcup \{\mathcal{D}^{(t, \cdot)}; t \in \mathcal{R}\}$.

A real-valued function f with domain $f = G$ open in $\mathcal{R}$ is said to be Newton-integrable, and we will write $f \in \mathcal{N}$, iff there exists a function F such that $F'(t) = f(t)$ for all $t \in G$; a similar definition will also be assumed for $\mathcal{R}^n$-valued functions, and for closed interval domains $[\tau, \tau']$. In the latter case, the number $F(\tau') - F(\tau)$ is called the Newton integral of f over $[\tau, \tau']$, and denoted by $(\mathcal{N})\int_{\tau}^{\tau'} f$. For convenience, the Lebesgue integral may be denoted by $(\mathcal{L})\int$, and the set of Lebesgue integrable functions by $\mathcal{L}$.

Further, let c be a real-valued function on an interval $I = [\tau, \tau + \alpha]$. We write $c \in \mathcal{N}_m$ iff $c\varphi \in \mathcal{N}$ on I for each continuous $\varphi : I \to \mathcal{R}$. Evidently $\mathcal{N}_m \subset \mathcal{N}$.

3. <u>Definition</u>. Let $f : \mathcal{D} \to \mathcal{R}^n$ with $\mathcal{D}$ an open subset of the (t, x)-space. It will then be said that a $\varphi : [\tau, \tau + \alpha] \to \mathcal{R}^n$ is an $\mathcal{N}$-solution

of the differential equation (1.1) iff (1.2) holds everywhe re on $[\tau, \tau+\alpha]$ (with one-sided derivatives at the endpoints).

Remark. Evidently this requirement is equivalent to $\varphi(t) = \varphi(\tau) + (\mathcal{N})\int_\tau^t f(s, \varphi(s))$ for all $t \in [\tau, \tau+\alpha]$.

4. It would be most interesting to describe large classes of f's for which $\mathcal{N}$-solutions always exist. In analogy with the theory developed in [3] for, say, Perron integration, it seems natural to consider differential equations (1.1) such that

(4.1) for each $x \in proj_x \mathcal{D}$, $f(\cdot, x) \in \mathcal{N}$ on $\mathcal{D}^{(\cdot, x)}$;

(4.2) for each $t \in proj_t \mathcal{D}$, $f(t, \cdot)$ is continuous on $\mathcal{D}^{(t, \cdot)}$;

under some further boundedness conditions, e.g., that

(4.3) there exist m, $M \in \mathcal{N}$ on $proj_t \mathcal{D}$ such that $m(t) \leq f(t,x) \leq M(t)$ for each $(t, x) \in \mathcal{D}$.

However, we do not have any existence result in this direction: the usual reasoning via the Schauder fixed-point theorem fails here since it is not known whether, under the conditions exhibited, $f(t, \varphi(t)) \in \mathcal{N}$ for each continuous φ. The present paper is then devoted to the special case of linear equations.

5. In what follows, we shall be concerned with linear differential equations of the form

$$x' = A(t)x + b(t) \tag{5.1}$$

under the assumption that the $n \times n$ matrix $A = (a_{ij})$ and the $n \times 1$ matrix $b = (b_i)$ have

(5.2) $\quad a_{ij}\,,\ b_i \in \mathcal{N} \quad$ for $1 \leq i, j \leq n$

on a fixed closed interval $I = [\tau, \tau + \alpha]$. (This conforms to the general situation from Item 3 after extending A, b appropriately to some open interval containing I.)

The first question is the existence problem: Given an initial value $\xi \in \mathcal{R}^n$, does there exist an $\mathcal{N}$-solution of (5.1) on I such that $\varphi(\tau) = \xi$. For this we have the two results described in Items 8 and 11. The present notation and assumptions will be preserved.

6. Definition. A function $a : I \to \mathcal{R}$ is called $\mathcal{N}$-semibounded on I iff there exists a function $c : I \to \mathcal{R}$ such that $c \in \mathcal{N}_m$ and

(6.1) either $a(t) \geq c(t)$ for $t \in I$, or $a(t) \leq c(t)$ for $t \in I$.

In this case a may be termed $\mathcal{N}$-semibounded by c.

Similarly, an $n \times n$ matrix $A = (a_{ij})$ of functions a_{ij} on I is called $\mathcal{N}$-semibounded on I iff there exists a matrix $C = (c_{ij})$, $n \times n$, of functions c_{ij} such that each a_{ij} is $\mathcal{N}$-semibounded by c_{ij}.

7. Lemma. Let $a \in \mathcal{N}$ be $\mathcal{N}$-semibounded on I. Then $a \in \mathcal{N}_m$.

Proof. Let a be $\mathcal{N}$-semibounded by c, e.g., $a \geq c$; then $a - c \geq 0$ and $a - c \in \mathcal{N}$. Let $\varphi : I \to \mathcal{R}$ be continuous. Now put $E(t) = (\mathcal{N})\int_\tau^t (a-c)$;

then E non-decreases on I, so that the Riemann-Stieltjes integral $F(t) = \int_\tau^t \varphi\, d\, E$ exists for each $t \in E$. Now it suffices to show that $F'(t) = \varphi(t)(a(t) - c(t))$ for all $t \in I$, since then the assertion follows from $c\varphi \in \mathcal{N}$. Indeed, we have e.g. for each $\tau \leq t < \tau + \alpha$ and each (small) $h > 0$

$$\min_{[t,t+h]} \varphi \cdot \frac{1}{h}(E(t+h) - E(t)) \leq \frac{1}{h}(F(t+h) - F(t)) =$$

$$= \frac{1}{h}\int_t^{t+h} \varphi\, d\, E \leq \max_{[t,t+h]} \varphi \cdot \frac{1}{h}(E(t+h) - E(t)) ,$$

thus completing the proof.

8. <u>Theorem</u>. Consider (5.1) and an initial value $\xi \in \mathcal{R}^n$, and assume (5.2) and also

(8.1) each a_{ij} is Lebesgue integrable over I ;

(8.2) A is $\mathcal{N}$-semibounded on I.

Then there exists an $\mathcal{N}$-solution φ of (5.1) on I with $\varphi(\tau) = \xi$.

<u>Proof</u>. Put $\varphi_0 \equiv \xi$, and then, for each $i = 0, 1, \ldots$

(8.3) $$\varphi_{i+1}(t) = \xi + (\mathcal{L})\int_\tau^t A\varphi_i + (\mathcal{N})\int_\tau^t b .$$

Now set $\beta(t) = (\mathcal{L})\int_\tau^t \max_k \sum_{j=1}^{m} |a_{kj}(t)|$; then by induction one easily obtains the estimate

$$|\varphi_{i+1}(t) - \varphi_i(t)| \leq \frac{1}{i!} \max_I |\varphi_1 - \varphi_0| (\beta(t))^i .$$

In particular, the series $\varphi = \varphi_0 + (\varphi_1 - \varphi_0) + \ldots$ converges uniformly on I. Hence and from (8.3),

(8.4) $$\varphi(t) = \xi + (\mathcal{L})\int_\tau^t A\varphi + (\mathcal{N})\int_\tau^t b$$

for all $t \in I$; and here, according to Lemma 7, $(\mathcal{L})$ may be replaced by $(\mathcal{N})$, concluding the proof.

9. In the authors' opinion, it is an interesting open question whether condition (8.2), or even both (8.1),(8.2), might be omitted entirely.

We prove that the first case occurs provided

$$b \in \mathcal{L} \cap \mathcal{N} . \tag{9.1}$$

It is then possible to write (8.4) in the form

$$\varphi(t) = \xi + (\mathcal{L})\int_\tau^t A\varphi + (\mathcal{L})\int_\tau^t b \tag{9.2}$$

so that φ is absolutely continuous on I. Now it suffices to prove that $a \in \mathcal{L} \cap \mathcal{N}$, φ absolutely continuous on I imply $a\varphi \in \mathcal{N}$ on I.

Put $F(t) = \int_\tau^t a$, $t \in I$, and let $G : I \to \mathcal{R}$ be defined as $G(t) = \int_\tau^t \varphi \, dF$, with the integration taken in the sense of Riemann-Stieltjes. We prove that

$$G'(t) = a(t)\varphi(t), \quad t \in I . \tag{9.3}$$

Fix $t \in I$ and put $H(x) = F(x) - F(t) - (x-t)a(t)$, $x \in I$.

Then $\int_t^{t+h} \varphi \, dF = \int_t^{t+h} \varphi \, dH + a(t)\int_t^{t+h} \varphi$. Now, it is easily seen that, to prove (9.3), it is sufficient to show that

$$\lim_{h\to 0} h^{-1} \int_t^{t+h} \varphi \, dH = 0 . \tag{9.4}$$

Using

$$\int_t^{t+h} \varphi \, dH = [\varphi H]_t^{t+h} - \int_t^{t+h} H \, d\varphi \tag{9.5}$$

the required result follows from $H(t) = 0$ and $H'(t) = \lim_{h\to 0} h^{-1} H(t+h) = 0$.

10. On the other hand, for equations of a slightly more special form, one can omit both (8.1) and (8.2). This concerns equations (5.1) in $2n$ -spaces with coefficient matrix

$$\begin{pmatrix} 0, & E \\ A, & 0 \end{pmatrix} \tag{10.1}$$

where the displayed submatrices all have type $n \times n$; or, in another formulation, second-order equations in n -space of the form

$$x'' = A(t)x \quad \text{for} \quad t \in I . \tag{10.2}$$

The proof of the following theorem is based on a private communication by J. Mařík to one of the authors concerning the case $n = 1 = \xi$, $\eta = 0$.

11. <u>Theorem</u>. Consider (10.2) and initial values ξ, η in $\mathcal{R}^n$, and assume $A \in \mathcal{N}$. Then there exists an $\mathcal{N}$ - solution φ of (10.2) on I with $\varphi(\tau) = \xi$, $\varphi'(\tau) = \eta$.

<u>Proof</u>. First set $B(t) = (\mathcal{N})\int_\tau^t A$ for $t \in I$; since B is continuous on I and $B(\tau) = 0$, one may choose a $\sigma > 0$ such that

$$\max_{t \in [\tau, \tau+\sigma]} |B(t)| < \frac{1}{3\sigma} . \tag{11.1}$$

Now define $\psi_0 \equiv 0$ and, for $i = 0, 1, \ldots$

$$\psi_{i+1}(t) = B(t)(\xi + \int_\tau^t \psi_i) + \eta - \int_\tau^t B\psi_i . \tag{11.2}$$

Thus

$$\psi_{i+1}(t) - \psi_i(t) = B(t)\int_\tau^t (\psi_i - \psi_{i-1}) - \int_\tau^t B(\psi_i - \psi_{i-1})$$

and therefore, using (11.1) (with $\| \cdots \|$ denoting maxima in $[\tau, \tau+\sigma]$),

$$\|\psi_{i+1} - \psi_i\| \leq \frac{1}{3\sigma}\|\psi_i - \psi_{i-1}\|\sigma + \frac{1}{3\sigma}\|\psi_i - \psi_{i-1}\|\sigma =$$

$$= \frac{2}{3}\|\psi_i - \psi_{i-1}\| .$$

It follows that the sequence $\{\psi_i\}$ converges uniformly in $[\tau, \tau+\sigma]$; let ψ denote its limit. Then from (11.2)

$$\psi(t) = B(t)(\xi + \int_\tau^t \psi) + \eta - \int_\tau^t B\psi \quad . \tag{11.3}$$

Finally, set $\varphi(t) = \xi + \int_\tau^t \psi$; then $\varphi' = \psi$ and $\varphi''(t) = B'(t)(\xi + \int_\tau^t \psi) + B(t)\psi(t) - B(t)\psi(t) = A(t)\varphi(t)$ and also $\varphi(\tau) = \xi,\ \varphi'(\tau) = \psi(\tau) = B(\tau)\xi + \eta = \eta$.

This establishes the existence of the required $\mathcal{N}$ -solution at least on a subinterval $[\tau, \tau+\sigma]$. Now, it is easy to see that the solution may be extended over the whole interval $[\tau, \tau+\alpha]$ in the customary manner.

12. <u>Theorem</u>. Under the assumptions of Theorem 8, equation (5.1) has unicity of $\mathcal{N}$ -solutions to arbitrary initial conditions.

<u>Proof</u>. As in all linear problems, it suffices to show that if an $\mathcal{N}$ -solution φ of

$$x' = A(t)x \tag{12.1}$$

has $\varphi(\tau) = 0$, then $\varphi \equiv 0$ in some neighborhood of τ . Consider the "fundamental" solution Y of the matrix equation

$$Y' = -A^T(t)Y, \quad Y(t) = E$$

(where A^T is the transpose of A and E the unit matrix).

Since $-A^T$ satisfies with A the assumptions of Theorem 8, such a solution Y indeed exists.

Now,

$$(Y^T \varphi)' = -(Y^T A)\varphi + Y^T (A\varphi)$$

so that

$$Y^T(t)\varphi(t) = const = Y^T(\tau)\varphi(\tau) = 0 . \tag{12.2}$$

Also, $Y(t)$ is continuous, and non-singular at $t = \tau$, so that it remains non-singular on $[\tau, \tau + \sigma]$ for small $\sigma > 0$, whereupon (12.2) yields $\varphi \equiv 0$ near τ.

13. Actually, Theorem 12 is a rather special case of a result, asserting that existence of solutions (for "negative time") of the adjoint problem implies unicity (for "positive time") of the original linear problem [2]. The same result then applies to the situation treated in Theorem 11; it suffices to observe that the adjoint equation to that determined by (10.1) has coefficient matrix

$$\begin{pmatrix} 0, & -A^T \\ -E, & 0 \end{pmatrix}$$

and this is again reducible to the form (10.2).

R e f e r e n c e s

[1] C. CARATHÉODORY: Vorlesungen über reelle Funktionen, Leipzig 1927.

[2] O. HÁJEK: Theory of processes I, Czech. Math. Journal (to appear).

[3] K. KARTÁK: A generalization of the Carathéodory theory of differential equations, Czech.Math.Journal (to appear).

[4] S. SAKS: Theory of the Integral, Warszawa 1937.

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