Commentationes Mathematicae Universitatis Carolinae

Jiří Sichler

Concerning endomorphisms of finite algebra

Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 3, 405--414

Persistent URL: http://dml.cz/dmlcz/105122

Commentationes Mathematicae Universitatis Carolinae
8,3(1967)

CONCERNING ENDOMORPHISMS OF FINITE ALGEBRA

J. SICHLER, Praha

Consider an (universal) algebra A on a finite set X and the semigroup $H(A)$ of all its endomorphisms. It was proved in [1] that not every transformation semigroup containing the identity mapping is equal to $H(A)$ for some A.

The aim of the present note is to prove the following:

If X has a cardinality greater than 4 and if every permutation of X belongs to $H(A)$, then either $H(A)$ consists exactly of all the permutations and all the constant mappings, or $H(A)$ is the full transformation semigroup on X.

This result immediately implies a finite analogon of the counterexample 2 in [1].

First, some notation and definitions.

As usually, an ordinal number $\varkappa$ is defined as the set of all the ordinals less than $\varkappa$.

If X, Y are sets, we denote by X^Y the set of all the mappings $F: Y \to X$. The cardinal number of the set X will be denoted by $|X|$.

If k is an ordinal number and if X is a set, then every mapping $\omega : X^{\varkappa} \to X$ will be called a $\varkappa$-ary algebraic operation on X. ω is termed a projection on $M \subseteq X^{\varkappa}$, if

$$(\exists \lambda \in \varkappa)(\forall \varphi \in M)(\omega(\varphi)) = \varphi(\lambda)) ,$$

ω is termed a quasiprojection on $M \subseteq X^{\varkappa}$, if

$$(\forall \varphi \in M)(\exists \lambda \in \varkappa)(\omega(\varphi)) = \varphi(\lambda)) .$$

An algebra is a couple $\langle X, \Omega \rangle$, where X is a set and Ω is some set of operations on X .

Denote by $H\langle X, \Omega \rangle$ the set of all the endomorphisms of the algebra $\langle X, \Omega \rangle$.

It is easy to see that if every $\omega \in \Omega$ is a projection, then $H\langle X, \Omega \rangle = X^X$.

As

$$(1) \quad H\langle X, \Omega \rangle = \bigcap_{\omega \in \Omega} H\langle X, \{\omega\} \rangle ,$$

we will consider algebras with one operation only.

We write $\langle X, \omega \rangle$ instead of $\langle X, \{\omega\} \rangle$.

For any set X , put

$$\mathcal{P} = \{F \in X^X \mid F \text{ is } 1-1 \text{ onto}\} ,$$

$$\mathcal{C} = \{F \in X^X \mid |F(X)| = 1\} .$$

If $\varkappa$ is an ordinal, put

$$\mathcal{F}_2 = \{\varphi \in X^\varkappa \mid |X - \varphi(\varkappa)| \geq 2\} ,$$

$$\mathcal{K} = \{\varphi \in X^\varkappa \mid \varphi \text{ is } 1-1 \text{ onto}\} .$$

If $|X| = \varkappa = n$ is finite, put

$$[k, n-k] = \{G \in X^X \mid (\exists g, h)(g \neq h, G(X) = \{g, h\}, |G^{-1}(g)| = k)\} ,$$

$$(k, n-k) = \{\varphi \in X^n \mid (\exists a, b)(a \neq b, \varphi(n) = \{a, b\}, |\varphi^{-1}(a)| = k)\}$$

for any positive integer $k \leq \frac{n}{2}$.

<u>Lemma 1</u>. Let $\langle X, \omega \rangle$ be an algebra, $|X| > 2$,

$\mathcal{P} \subseteq H\langle X, \omega\rangle$. Then $\mathcal{C} \subseteq H\langle X, \omega\rangle$.

Proof. Let $\omega : X^{\varkappa} \longrightarrow X$. For any $x \in X$ let $\varphi_x \in X$ be such that $\varphi_x(\varkappa) = \{x\}$. We define $F \in X^X$ by $F(x) = \omega(\varphi_x)$. For any $P \in \mathcal{P}$ we have $F \circ P = P \circ F$ so that (as $|X| > 2$) $\omega(\varphi_x) = F(x) = x$ for any $x \in X$. This is equivalent with the assertion of the lemma.

Lemma 2. Let $\langle X, \omega\rangle$ be an algebra, $|X| > 1$ finite. Let $\mathcal{P} \subseteq H\langle X,\omega\rangle$, $H\langle X,\omega\rangle \cap (X^X - (\mathcal{P} \cup \mathcal{C})) \neq \emptyset$. Then there is a $G \in H\langle X, \omega\rangle$ such that $|G(X)| = 2$.

Proof. Let $F \in H\langle X,\omega\rangle \cap (X^X - (\mathcal{P} \cup \mathcal{C}))$, $F(X) = \{a_1, \dots, a_k\}$. Put $A_i = F^{-1}(a_i)$ $(i = 1, \dots, k)$. As $|F(X)| < n$, there are an $i_0 \in \{1, \dots, k\}$ and $a, b \in A_{i_0}$ such that $a \neq b$. There exists a $P \in \mathcal{P}$ such that $P(a_{i_0}) = a$, $P(a_1) = b$, $P(a_i) \in A_i$ for any $i \neq 1$, $i \neq i_0$. Evidently, $|(F \circ P \circ F)(X)| = |F(X)| - 1$, $F \circ P \circ F \in H\langle X, \omega\rangle$. The conclusion follows by induction.

Lemma 3. Let $\langle X, \omega\rangle$ be an algebra, $\omega : X^{\varkappa} \longrightarrow X$, $\mathcal{P} \subseteq H\langle X,\omega\rangle$. Then ω is a quasiprojection on $\mathcal{F}_2$ and a projection on $\mathcal{K}$.

The proof is easy.

Remark. If $\omega : X^2 \to X$, $|X| \geq 4$, $\mathcal{P} \subseteq H\langle X, \omega\rangle$, then ω is a projection on X^2 .

Lemma 4. Let $\langle X, \omega\rangle$ be an algebra, $|X| = n$, $\omega : X^n \to X$. Then

a) If $G \in [k, n-k]$, $\{G\} \cup \mathcal{P} \subseteq H\langle X,\omega\rangle$, then ω is a projection on $\mathcal{K} \cup (k, n-k)$, $[k, n-k] \subseteq H\langle X, \omega\rangle$.

b) If $[k, n-k] \subseteq H\langle X, \omega\rangle$, then $\mathcal{P} \subseteq H\langle X, \omega\rangle$.

c) If $[1, n-1] \subseteq H\langle X, \omega\rangle$, then ω is a quasiprojection on X^n.

The proof is obvious.

Lemma 5. Let $\langle X, \omega\rangle$ be an algebra, $|X| = n$, $\omega : X^n \to X$.

If $[k, n-k] \subseteq H\langle X, \omega\rangle$, then ω is a projection on

$$\mathcal{K} \cup \bigcup_{\ell \geq k} (\ell, n-\ell).$$

Proof. Let $\ell > k$, $\varphi \in (\ell, n-\ell)$, $\varphi(n) = \{a, b\}$, $|\varphi^{-1}(a)| = \ell$. Let $\psi \in X^n$ be arbitrary but fixed such that

ψ is one-to-one on $\varphi^{-1}(a)$,

$\psi(i) \neq b$ whenever $i \in \varphi^{-1}(a)$,

$\psi(i) = b$ whenever $i \in n - \varphi^{-1}(a)$.

As $1 \leq k$ and $\ell < n-k$, there is an $F \in [k, n-k]$ such that $F(\psi(i)) = a$ for any $i \in \varphi^{-1}(a)$, $F(b) = b$. For any such F we have $F \circ \psi = \varphi$.

For any $I \subseteq \varphi^{-1}(a)$ such that $|I| = k$ define a mapping $G_I \in X^X$ as follows:

$$\text{(3)} \qquad \begin{cases} G_I(\psi(i)) = a & \text{if } i \in I, \\ G_I(x) = b & \text{otherwise}. \end{cases}$$

Evidently, $G_I \in [k, n-k]$, $G_I \circ \psi \in (k, n-k)$.

By lemmas 3 and 4, there is an $s \in n$ such that $\omega(\chi) = \chi(s)$ for any $\chi \in (k, n-k)$.

We shall distinguish two cases.

I. $s \in \varphi^{-1}(a)$. Let us take a G_I (see (3)) with $s \in I$. Since $G_I \circ \psi \in (k, n-k)$ and $G_I \in H\langle X, \omega\rangle$, we obtain $G_I(\omega(\psi)) = a$. Thus, $\omega(\psi) = \psi(i)$ for some $i \in \varphi^{-1}(a)$. On the other hand, $\omega(\varphi) = \omega(F \circ \psi) = F(\omega(\psi)) = \varphi(i)$. As $s \in \varphi^{-1}(a)$, we have $\varphi(s) = a = \varphi(i)$.

II. $s \in n - \varphi^{-1}(a)$. ω is a quasiprojection on $\{\psi\}$. (By lemma 4 for $k = 1$; if $k > 1$, then $n - \ell \geq 3$ and the assertion follows from lemma 3.) Consequently, ω is a quasiprojection also on $\{\varphi\}$. Let us suppose that $\omega(\varphi) \neq \varphi(s)$, i.e. that $\omega(\varphi) = a$. Then $\omega(\psi) = \psi(i)$ for some $i \in \varphi^{-1}(a)$. Take a G_I with $i \in I$. Then $\omega(G_I \circ \psi) = G_I(\omega(\psi)) = a$; on the other hand, $G_I \circ \psi \in (k, n-k)$, so that $\omega(G_I \circ \psi) = G_I(\psi(s)) = G_I(b) = b$. This is a contradiction.

Lemma 6. Let $|X| = 5$, $\omega: X^5 \to X$, $[k, 5-k] \subseteq H\langle X, \omega\rangle$ for some k. Then ω is a quasiprojection on X^5.

Proof. By lemma 4, it suffices to prove this for the case k = 2, by lemma 3, it suffices to prove the assertion for $\varphi \in X^5$ such that $|\varphi(5)| = 4$. Thus, let $\{a\} = X - \varphi(5)$, let $\varphi(m) = \varphi(n)$ $(m \neq n)$ and let $\omega(\pi) = \pi(s)$ for any $\pi \in \mathcal{K} \cup (2, 3)$.

We shall distinguish two cases.

I. $t \neq s \implies \varphi(t) \neq \varphi(s)$. Let us define a mapping $F \in X^X$ as follows: $F(\varphi(m)) = F(a) = a$, $F(x) = \varphi(s)$ otherwise. Evidently $F \in [2, 3]$, $F \circ \varphi \in (2, 3)$, $F(\omega(\varphi)) = \omega(F \circ \varphi) = F(\varphi(s)) = \varphi(s)$ by lemma 4. Thus, $\omega(\varphi) \neq a$.

II. $n = s$, $\varphi(m) = \varphi(s)$. Let $\{i_1, i_2, i_3\} = 5 - \{m, n\}$. Put $G(a) = G(\varphi(i_1)) = G(\varphi(i_2)) =$

$= a, \quad G(\varphi(m)) = G(\varphi(i_3)) = \varphi(m)$.

Evidently $G \in [2,3]$, $G \circ \varphi \in (2,3)$. If $\omega(\varphi) = a$, then $a = G(a) = G(\omega(\varphi)) = \omega(G \circ \varphi) = G(\varphi(b)) = G(\varphi(m)) = \varphi(m)$, which is a contradiction.

Lemma 7. Let $n \geq 5$, $n \geq 2k$, $k > l > 0$. Then there are $n_1 > 0$, $n_2 > 0$ such that $l + 2n_1 + n_2 = n$, $l + n_1 = k$, $n_1 + n_2 = n - k$. Moreover, if $n > 5$, then $n_1 \geq 2$ or $l + n_2 \geq 4$.

Proof. Put $n_1 = k - l$, $n_2 = n + l - 2k$.

Lemma 8. Let $\langle X, \omega \rangle$ be an algebra, let $|X| = n \geq 5$, $\omega : X^n \to X$.
If $[k, n-k] \subseteq H\langle X, \omega \rangle$, then ω is a projection on

$$\mathcal{K} \cup \bigcup_{l \neq k} (l, n-l) .$$

Proof. Let $k > l > 0$, $\varphi \in (l, n-l)$, $\varphi(n) = \{a, b\}$, $|\varphi^{-1}(a)| = l$, $X = \{a, a_1, \dots, a_{n-1}\}$.
Let n_1 and n_2 be the numbers from lemma 7. Let $A_1, A_2, A_3 \subseteq n - \varphi^{-1}(a)$ be disjoint sets with $|A_1| = |A_2| = n_1$, $|A_3| = n_2$. We have $A_1 \cup A_2 \cup A_3 = n - \varphi^{-1}(a)$.
As $n \geq 5$, we can define $\psi \in X^n$ as follows:

$$\psi(i) = a \quad \text{if} \quad i \in \varphi^{-1}(a),$$
$$\psi(i) = a_j \quad \text{if} \quad i \in A_j \qquad j = 1, 2, 3 .$$

Put

$$F(a_j) = b \quad \text{if} \quad j \in \{1, \dots, n-k\},$$
$$F(x) = a \quad \text{otherwise.}$$

Evidently $F \circ \psi = \varphi$.
As $k \geq 2$, $n-k > 2$, then there are mappings G_1, $G_2 \in [k, n-k]$ such that $G_1(a) = G_2(a) = G_1(a_1) =$

$= G_2(a_2) = a, \; G_1(a_3) = G_2(a_3) = G_1(a_2) = G_2(a_1) = b.$

Obviously, $G_1 \circ \psi, \; G_2 \circ \psi \in (k, n-k)$.

Lemmas 7, 3 and 6 yield that ω is a quasiprojection on $\{\psi\}$. By lemma 4 and our assumption, $\omega(\chi) = \chi(s)$ for any $\chi \in \mathcal{K} \cup (k, n-k)$.

Consider two cases.

I. If $s \in \varphi^{-1}(a)$, then $G_i(\omega(\psi)) = \omega(G_i \circ \psi) = G_i(\psi(s)) = G_i(a) = a$. As ω is a quasiprojection on $\{\psi\}$, we have $\omega(\psi) = a$. Further, $\omega(\varphi) = \omega(F \circ \psi) = F(\omega(\psi)) = F(a) = a$. As $s \in \varphi^{-1}(a)$, we obtain $\omega(\varphi) = \varphi(s)$.

II. Let $s \in n - \varphi^{-1}(a)$. If $s \in A_1 \cup A_3$, then $G_2(\omega(\psi)) = \omega(G_2 \circ \psi) = G_2(\psi(s)) = G_2(a_1) = b$. As ω is a quasiprojection on $\{\psi\}$, then $\omega(\psi) = a_1$ or $\omega(\psi) = a_3$. In both cases $\omega(\varphi) = b = \varphi(s)$. If $s \in A_2$, we use G_1 similarly.

Lemma 9. Let $\langle X, \omega \rangle$ be an algebra, $|X| = n \geq 5$, $\omega : X^n \to X$. Let $\mathcal{P} \subseteq H\langle X, \omega \rangle$. If $H\langle X, \omega \rangle \cap (X^X - (\mathcal{P} \cup \varphi)) \neq \emptyset$, then $H\langle X, \omega \rangle = X^X$.

Proof. By lemma 2 and lemma 4, $[k, n-k] \subseteq H\langle X, \omega \rangle$ holds for some k. Lemmas 4, 5, 8 yield that ω is a projection on $\mathcal{K} \cup \bigcup_{\ell} (\ell, n-\ell)$. We shall prove first that ω is a quasiprojection on X^n. By lemma 3 it suffices to prove this for $\varphi \in X^n$ such that $|\varphi(n)| = n-1$. Let $\{a\} = X - \varphi(n)$, let, for any $\psi \in \mathcal{K} \cup \bigcup_{\ell}(\ell, n-\ell)$, $\omega(\psi) = \psi(s)$ hold. The case $k = 1$ is proved in lemma 4. Let $k > 1$, $\omega(\varphi) = a$. Then there is an $F \in [k, n-k]$ such that $F(\varphi(s)) = \varphi(s)$, $F(a) = a$.

As $k > 1$, then $F \circ \varphi \in (\ell, n-\ell)$ for some ℓ. We obtain $\omega(F \circ \varphi) = F(\varphi(s)) = \varphi(s)$. $F(\omega(\varphi)) = F(a) = a$, on the other hand, however $F \in H\langle X, \omega\rangle$, which is a contradiction.

Let $\chi \in X^n$, $|\chi(n)| \leq n-1$, let $b \notin \chi(n)$.

I. Let $|X - \chi(n)| \geq k-1$. Then there is an $F \in [k, n-k]$ such that $F(\chi(s)) = \chi(s)$ and $F(\chi(i)) = b$ for $\chi(i) \neq \chi(s)$. There is $F \circ \chi \in (\ell, n-\ell)$ for some ℓ. We obtain $F(\omega(\chi)) = \omega(F \circ \chi) = F(\chi(s)) = \chi(s)$. As ω is a quasiprojection, $\omega(\chi) = \chi(s)$.

II. Let $|X - \chi(n)| < k-1$. Then $|\chi(n)| > n-k$. For any $N \subseteq \chi(n)$ with $|N| = n-k$ let us define a mapping $F_N \in X^X$ as follows:

$$F_N(x) = \chi(s) \quad \text{if } x \in N,$$
$$F_N(x) = b \quad \text{otherwise.}$$

Evidently, $F_N \in [k, n-k]$. If $\omega(\chi) = \chi(m) \neq \chi(s)$, then there is an $N \subseteq \chi(n)$ such that $|N| = n-k$, $\chi(m) \in N$, $\chi(s) \notin N$. Then $b = F_N(\chi(s)) = \omega(F_N \circ \chi)$, since $F_N \circ \chi \in (\ell, n-\ell)$ for some ℓ; on the other hand, $\omega(F_N \circ \chi) = F_N(\omega(\chi)) = F_N(\chi(m)) = \chi(s)$, which is a contradiction.

Lemma 10. Let $\langle X, \omega\rangle$ be an algebra, $\omega : X^{\varkappa} \to X$, $|X| = \alpha$. Then there are operations Ω_Φ ($\Phi \in N$) such that

a) $\Omega_\Phi : X^\alpha \to X$,

b) $H\langle X, \omega\rangle = H\langle X, \{\Omega_\Phi \mid \Phi \in N\}\rangle$.

Proof. I. Let $|\varkappa| > \alpha$. Put $N = \{\Phi \in \alpha^{\varkappa} \mid \Phi(\varkappa) = \alpha\}$, $N_\Phi = \{\varphi \in X^{\varkappa} \mid (\exists \psi \in X^\alpha)(\varphi = \psi \circ \Phi)\}$.

For any $\Phi \in N$ define $\Omega_\Phi : X^\alpha \to X$ by $\Omega_\Phi(\Psi) = \omega(\Psi \circ \Phi)$.

The evident relation $X^k = \bigcup_{\Phi \in N} N_\Phi$ and a direct computation prove our assertion.

II. The case of $|\mathscr{X}| \leqq \alpha$ is trivial.

<u>Theorem 1</u>. Let $\langle X, \Omega \rangle$ be a finite algebra such that $\mathcal{P} \subseteq H\langle X, \Omega \rangle$. Then the following assertions hold:

I. If $|X| = 2$, then either $H\langle X, \Omega \rangle = \mathcal{P}$ or $H\langle X, \Omega \rangle = X^X$.

II. If $|X| = 4$, then there are three possibilities

1) $H\langle X, \Omega \rangle = \mathcal{P} \cup \varphi$,
2) $H\langle X, \Omega \rangle = X^X - [1, 3]$,
3) $H\langle X, \Omega \rangle = X^X$.

III. If $|X| = 3$ or $|X| \geq 5$, then either $H\langle X, \Omega \rangle = \mathcal{P} \cup \varphi$ or $H\langle X, \Omega \rangle = X^X$.

<u>Proof</u>. According to (1) and lemma 10 we can admit an arbitrary set Ω of operations on X.

I. is trivial.

II. Let us define $\pi : X^3 \to X$ as follows: $\pi(\varphi) = a$ if $\{a\} = X - \varphi(3)$, $\pi(\varphi) = \varphi(0)$ if $\varphi(1) = \varphi(2)$, $\pi(\varphi) = \varphi(1)$ if $\varphi(0) = \varphi(2)$, $\pi(\varphi) = \varphi(2)$ if $\varphi(0) = \varphi(1)$. We see by direct computation that this is an example for the case 2). There are no other cases except of 1),2),3) (lemmas 1, 4, 5).

III. The case $|X| = 3$ is trivial, the case $|X| \geq 5$ follows from lemma 9.

<u>Notation</u>. For any $F \in X^X$ let us denote by π_F its partition. We use the following notation: $F \prec G$

indicates $\pi_F \supseteq \pi_G$ and $F(X) \subseteq G(X)$. The relation

$$F \prec G \ \& \ G \prec F$$

is an equivalence on X^X, the classes of which are $\mathcal{H}$-classes of the semigroup X^X (cf.[2]). Hence, the relation $\prec$ induces a partial ordering $\leqq$ on the set of all the $\mathcal{H}$-classes of X^X. If H is an $\mathcal{H}$-class, put

$$L(H) = \{\cup K \mid K \text{ is an } \mathcal{H}\text{-class}, K \leqq H\},$$
$$I(H) = f(X) \quad (f \in H).$$

Let us denote by $\mathcal{C}_H$ the set of all the $C \in \mathcal{C}$ such that $C(X) \subseteq I(H)$.

As an immediate corrolary of the Theorem 1 we obtain

<u>Theorem 2</u>. Let $\langle X, \Omega \rangle$ be an algebra, $H \subseteq X^X$ an $\mathcal{H}$-class such that $I(H)$ is finite and $|I(H)| = 3$ or $|I(H)| \geqq 5$. Let $H \subseteq H\langle X, \Omega \rangle$. Then

a) if H contains no idempotent, then

(4) $L(H) \subseteq H\langle X, \Omega \rangle$

b) if H contains an idempotent, then either (4) or $H\langle X, \Omega \rangle \cap L(H) = H \cup \mathcal{C}_H$.

I thank Z. Hedrlín and A. Pultr for the suggestion of the problem and for much valuable advice.

R e f e r e n c e s

[1] M. ARMBRUST, J. SCHMIDT: Zum Cayleyschen Darstellungssatz, Math. Annalen 154(1964), 70-72.

[2] A.H. CLIFFORD, G.B. PRESTON: The algebraic theory of semigroups, AMS 1961, Rhode Island.

(Received February 10, 1967)